CONVERTING AMONG FRACTIONS, DECIMALS AND PERCENTS: AN EXPLORATION OF REPRESENTATIONAL USAGE BY MIDDLE SCHOOL TEACHERS

A Dissertation

by

MICHAEL TAPFUMA MUZHEVE

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

August 2008

Major Subject: Curriculum and Instruction

CONVERTING AMONG FRACTIONS, DECIMALS AND PERCENTS: AN EXPLORATION OF REPRESENTATIONAL USAGE BY MIDDLE SCHOOL TEACHERS

A Dissertation

by

MICHAEL TAPFUMA MUZHEVE

Submitted to the Office of Graduate Studies of
Texas A&M University
in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

Approved by:

Chair of Committee,	Robert M. Capraro
Committee Members,	Mary M. Capraro
	Gerald Kulm
	J. Maurice Rojas
	Radhika Viruru
Head of Department,	Dennie Smith

August 2008

Major Subject: Curriculum and Instruction

ABSTRACT

Converting among Fractions, Decimals, and Percents: An Exploration of Representational Usage by Middle School Teachers. (August 2008)

Michael Tapfuma Muzheve, B.Sc., University of Zimbabwe

M. Phil., University of Zimbabwe;

M.S., Texas A&M University

Chair of Advisory Committee: Dr. Robert M. Capraro

Using both quantitative and qualitative data collection and analyses techniques, this study examined representations used by sixteen (n = 16) teachers while teaching the concepts of converting among fractions, decimals, and percents. The classroom videos used for this study were recorded as part of the Middle School Mathematics Project (MSMP). The study also compared teacher-selected and textbook representations and examined how teachers' use of idiosyncratic representations influenced representational choices on the number test by the teachers' five hundred eighty-one (N = 581) students.

In addition to using geometric figures and manipulatives, a majority of the teachers used natural language such as the words nanny, north, neighbor, dog, cowboy, and house to characterize fractions and mathematical procedures or algorithms. Coding of teacher-selected representations showed that verbal representations deviated from textbook representations the most. Some teachers used the words or phrases bigger,

smaller, doubling, tripling, breaking-down, and building-up in the context of equivalent fractions.

There was widespread use of idiosyncratic representations by teachers, such as equations with missing or double equal signs, numbers and operators written as superscripts, and numbers written above and below the equal sign. Although use of idiosyncratic representations by teachers influenced representational choices by students on the number test, no evidence of a relationship between representational forms and degree of correctness of solutions was found. The study did reveal though that teachers' use of idiosyncratic representations can lead to student misconceptions such as thinking that multiplying by a whole number not equal to 1 gives an equivalent fraction.

Statistical tests were done to determine if frequency of representation usage by teachers was related to the textbook, highest degree obtained by teacher, certification, number of years spent teaching mathematics, number of years teaching mathematics at grade level, number of hours completed on professional development related to their textbook, and total number of days spent on the Interagency Education Research Initiative (IERI) professional development. The results showed representation usage was related to all the above variables, except the highest degree obtained and the total number of days spent on the IERI professional development.

DEDICATION

To those who do not give-up easily, to those who fight till the end, to the airman, and therefore to my brother and friend Felix Mwarianesu Muzheve (07/07/1971-02/13/2008). Rest in peace Mukanya.

ACKNOWLEDGEMENTS

I would like to thank my committee chair, Dr. R. M. Capraro, and my committee members, Dr. M. M. Capraro, Prof. Kulm, Dr. Rojas, and Dr. Viruru, for their guidance and support throughout the course of this research. This research would never have been a success had they not provided the support, encouragement, constructive criticism, and lots help in whatever form it came. Special thanks to Dr. Robert Capraro for providing both scientific and moral support.

Thanks also go to my friends and colleagues and the department faculty and staff for making my time at Texas A&M University a great experience. I also want to extend my gratitude to the American Association for the Advancement of Science (AAAS) and the Middle School Mathematics Program (MSMP) who provided one of the instruments used in this study as well as permission to use their data.

Finally, thanks to my wife Vaidah for her patience, love, and encouragement. This acknowledgement would be greatly flawed if I did not thank my daughter Ruvimbo and my son Vimbainashe for their inspiration, inquisitiveness, and managing to keep me on my toes through their continuous assessment.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

INTRODUCTION

Background

According to the National Council of Teachers of Mathematics (NCTM, 2000), "At the heart of flexibility in working with rational numbers is a solid understanding of different representations for fractions, decimals, and percents" (p.215). As they solve problems, students are expected to consider advantages and disadvantages of various representations of quantities and they should be able to compare and translate among number forms to solve problems (NCTM, 2000). Engaging in mathematical argumentation and producing mathematical evidence requires that students talk or write in ways that expose their reasoning to their peers and to their teacher. These activities which are about communication and language need to be taught and learned in school classrooms (Lampert & Cobb, 2003).

My View of Representations and What Inspired This Study

My view of mathematics as a language greatly influences how I look at the way representations are used during instruction or in solutions to problems on home-works, tests, quizzes, exams or any situation in which someone else (sometimes the reader themselves) have to read and make sense of the work at a later time. Take for example the following numbers:

22/8 2 6/8 8/6 4/3 11/4 2 3/4

This dissertation follows the style of *Middle Grades Research Journal*.

What can be said about the above numbers? Well 22/8= 2 6/8 and 8/6= 4/3. The same is true for 11/4 and 2 ¾, but that is not all. Inserting any of the symbols ≤, ≥, ÷, ×, <, >, or ± in-between the three pairs of numbers would result in statements that all make sense mathematically. As one professor put it, writing numbers without relating them is like trying to construct a sentence without a verb. Encountering representations like the one I showed above when I was collecting data for one of my graduate class assignments prompted me to want to explore use of verbal and visual representations by middle school teachers. It is worth pointing out that the omission of the equal sign such as in the expression 8/6 4/3 means we do not have an equation, but rather two representations which are both in the class of verbal representations to which numbers belong.

While the correct use of representations might not always lead to positive learning outcomes (however that is measured), I strongly believe that teachers should strive to use representations correctly. I subscribe to the idea that representations are the language of mathematics (Coulombe & Berenson, 2001) and to the idea that mathematics is a universal language which any knowledgeable person should be able to read and understand without the need of an interpreter. Many times in the spirit of helping students and trying to make concepts accessible to students, teachers can instead negatively impact current and future student conceptions about certain concepts. Often teachers try to relate what is being learned to contexts that students are familiar with. For example, as I watched videos while collecting data for my class assignment, I heard teachers use phrases such as *next door neighbor*, *one from being a whole or from each other, consecutive* when they were referring to fractions like ½, 2/3, ¾, or 11/12. I thought it would be worthwhile to study the different representations that middle school mathematics teachers use in their classrooms as well as look at how that may impact learning of other concepts.

In part this study was also influenced by the fact that some of the representations I encountered as I worked on my class assignment were new to me having been educated in an education system different from the American system. I will end this section by drawing a parallel between learning to use mathematical representations and language development. I believe babies listen to everything their parents say, and store it away at an incredible rate. Instead of using "baby" words, I think parents should teach them the correct names for people, places and things.

Definitions

A representation is a configuration that depicts something else in some manner (Goldin, 2002). Representations can be divided into two categories (1) early representations and (2) mathematical representations (Capraro & Capraro, 2006). Mathematical representations are formal, standard, and internationally understood representations used to communicate mathematically. Classes of mathematical representations include diagrams, graphical displays, and symbolic expressions (NCTM, 2000). Early representations help make abstract mathematical concepts more approachable to students although they may not necessarily be intrinsically mathematical. Examples of classes of early representation include counters, pictures, imagery, drawings, cut-outs, micro-worlds and beans. Because of the lack of agreement on the best early representation(s) or procedure(s) for using it to ensure students have the ability to transition to mathematical representation and the fact that the choice and use of early representations is at the discretion of the teacher, early representations are often referred to as idiosyncratic representations (Capraro & Capraro, 2006). In this study, just like in Capraro's study, the term idiosyncratic representation indicates "a foundational classification of representations used by teachers or students in early stages of mathematical conceptualization" (p. 2). The term is also taken to refer to representations that are not typically found in mathematics textbooks, but ones that teachers or students use. These may be slight deviations from representations that one would find in mathematics textbooks. An example of one such representation is $4^{\div 2}/6^{\div 2} = 2/3$ in which the division symbols and the 2's are written as superscripts instead of writing the equation as $\frac{4 \div 2}{6 \div 2} = \frac{2}{3}$.

The phrase "written symbols" refers to both the mathematical symbols and the written words associated with them. Examples include ¼, 2/8, 0.25, 25% and *one-fourth, two-eights, twenty-five hundredths, and twenty-five percent* (Clement, 2004).

Representations have been discussed from the perspective of internal and external representations. The terms signified (internal) and signifiers (external) have also been used by some researchers in discussions about representations. Internal representation refers to unobservable mental configurations of individuals which are often inferred from what individuals say or do (Goldin & Kaput, 1996). To understand internal representations, researchers watch people performing cognitively demanding tasks and collect data for analysis. Types of data frequently used are response latencies, eye fixations, verbal reports, sorting, and free recall (Gagné, Yeckovich, & Yeckovich, 1993). Internal representations can also be thought of as abstractions of mathematical ideas or cognitive schemata that are developed by the learner through experience. These internal representations are stimulated by external representations which serve as conventions for the internal abstractions (Pape & Tchoshanov, 2001). External representation refers to physical, embodied, observable configurations, such as words, numerals, graphs and algebraic equations which allow us to discuss mathematical relations and meaning (Goldin & Kaput, 1996). An external representation has also been defined as something that stands for, symbolizes or represents objects and/or processes (Rosengrant, Heuvelen, & Etkina, 2005). Drawings, notes, equations, tables and Cartesian graphs are examples of standard external representational forms (Greeno & Hall, 1997).

Purpose of Study

The purpose of this study was to explore external representations (verbal and non-verbal) that middle school teachers used when explaining converting among fractions, decimals, and percents. I traced how various student understandings that may emerge as a result of representations they encounter when they learn these concepts can affect future or related learning. In particular, I examined misconceptions that can arise from the use of teacher's idiosyncratic representations.

Significance of Study

The assertion that "middle school is a critical leverage point for education reform efforts" (Kulm, Rosen, & Treistman, 1999, p.4) and the fact that representations can be considered as a language of mathematics (Coulombe & Berenson, 2001) made me feel that studying representation usage by middle school mathematics teachers can provide an insight into how teaching and learning can made more effective at middle school. While there has been extensive research on student misconceptions especially on the concept of equality (e.g., Baroody & Ginsburg, 1983; Capraro, Kulm, & Capraro, 2005; Ding, Li, Capraro, & Capraro, 2007; Kieran, 1981; Knuth, Alibali, McNeil, Weinberg, & Stephens, 2004; Knuth, Stephens, McNeil, & Alibali, 2006; Mack, 1995; McNeil et al., 2006) and how to diagnose and correct these misconceptions (e.g. Ashlock, 2006), a review of literature suggests not much has been said about the role played by teachers in the creation of student misconceptions particularly in middle grades. Cooney and Wilson (1993), argued one can gain a better understand of the broader domain of teachers' mathematical thinking and its influence on teaching and learning by considering teachers' thinking about a specific mathematical topic, such as fractions. In a study that focused on how students learn fractions, Thompson and Saldanha (2003) observed that "how students understand a concept has important implications for what they subsequently can do and learn." Analyses of what students learn should therefore trace "the implications that various understandings have for related or future learning" (p. 95).

This study focused on middle school teachers' explanations about converting among fractions, decimals, and percents as was revealed by the different representations (verbal and non-verbal) they used when teaching the concepts. The study also revealed how students' present and future understanding of mathematical concepts can be influenced by representations they encounter in middle school. This study also informed teacher education in terms of the kind of content and pedagogical knowledge teachers may need to gain through pre-service or in-service training in order to minimize student misconceptions that may arise when students are taught about converting among fractions, decimals, and percents.

Research Questions

1) What types of external representations (verbal or visual) of fractions and mathematical procedures or algorithms do middle school teachers use in lessons on converting among fractions, decimals, and percents? How frequent is the use of such representations?

2) What is the role of idiosyncratic representational forms in student solutions as evidenced on their post number tests? In particular,

   a. To what extent are the verbal and visual representations of fractions and corresponding mathematical procedures used by teachers and students similar to the representations in the textbook?

   b. To what extent are the representations used by teachers in the classroom similar to students' representations on the number test?

   c. Do student representational choices reflect teacher idiosyncratic representational forms?

   d. Do students' representational forms correspond to various degrees of correct solutions?

3) Are there differences in the teachers' representational choices (mathematical or idiosyncratic) for fractions and mathematical procedures as a result of teachers'

years of experience, level of education, type of certification or other emergent factors based on the quantification of the qualitative data obtained for question 1?

4) Can enacted student misconceptions on the number test be linked to idiosyncratic representations of fractions and mathematical procedures used by teachers?

Outline of Study

Mixed methods research techniques and methods were used in this study. Data used in this study was collected as part of the Middle School Mathematics Project (MSMP) which started in 2002 and ended in 2007. One set of data were collected through watching some of the classroom videos that were recorded during the 2002-03, 2003-04, and 2004-05 school years. Additional data were obtained from the MSMP data base in teacher characteristics and student performance and by examining students' posttest scripts.

Limitations of Study

According to Lincoln and Guba (1985), prolonged engagement, persistent observation, and triangulation can increase the probability that credible findings will be produced. They add that " the member check, whereby data, analytic categories, interpretations, and conclusions are tested with members of those stake holding groups from whom the data were originally collected, is the most crucial techniques for establishing credibility" (p. 314). Because the qualitative data used in this study is already on video tapes and because the project for which the data were originally collected already came to its conclusion, it will not be possible to achieve prolonged engagement and do member checks, although an IRB to do member checks outside of the original project could be obtained. The use of extant data meant I was sometimes left to speculate without a chance of asking follow-up questions in order to get clarifications or understand what the teachers were thinking.

According to Lincoln and Guba (1985) data recording modes vary along two dimensions; *fidelity* and *structure*. Video can be used to obtain fidelity which they define as the “ability of the investigator later to reproduce exactly the data as they become more evident to him or her in the field” (p.240). The authors cautioned though that video only records what the investigator chooses to record. This aspect of video-taping impacted this study in that data that would have been useful for this study was not captured on some of the videos. An example would be one classroom video in which the video camera did not capture what the teacher was writing on the board for most of the lesson.

As part of the study I used student number posttests. Nine of the sixteen items tests were multiple-choice questions that did not require students to show their work. This meant some of the students’ thought processes were not revealed. A better way to do it would have been to create an instrument after studying the representations that the teachers were using. In particular I would have designed an instrument requiring that students show all work. That way I believe I could have gained a better understanding of how the representations that the teachers use impact students’ understandings.

LITERATURE REVIEW

In this section I review literature related to my study that talks about origins and characterizations of student misconceptions and teacher knowledge. In particular, I review how students' mathematical thinking, representations and other instructional materials, limited learning experiences and instruction in general can lead to student misconceptions.

Origins and Characterizations of Student Misconceptions

The term student conceptions encompass the categories of student's beliefs, theories, meanings and explanations. When these conceptions are different or in conflict with corresponding taught or expert concepts, the term misconception is often used. A wide variety of terms have been generated to characterize student conceptions deemed to be different from expert or taught concepts (Confrey, 1990; Perkins & Simmons, 1988; Smith, diSessa, & Roschelle, 1993-1994).

Real-life Experiences and Student Conceptions

Studies have shown that children construct and bring to classroom instruction mathematics knowledge related to real life out-of-school situations. The mathematics may occasionally resemble that of formal instruction and whether it is correct or incorrect, students may draw on it in trying to solve problems posed in the contexts of real-life situations familiar to them. This applied, real-life, circumstantial knowledge which is generally assumed not to be a direct consequence of instruction is referred to as *informal knowledge* (Carpenter & Moser, 1984; Leinhardt, 1988; Mack, 1990; Saxe, 1988), *situated knowledge* (Brown, Collins, & Duguid, 1989), or *intuitive knowledge* (Leinhardt, 1988). Leinhardt's definition of intuitive knowledge also included knowledge that "has been highly altered since its school acquisition" (p.120). This type of knowledge is not normally learned from the teacher or from texts. Because it is highly contextualized, informal or intuitive knowledge is very hard to teach from or build on instruction because students tend to over generalize some concepts (Leinhardt, 1988; Mack, 1995). Contrary to these assertions, Carpenter and Moser (1984) bemoaned how the curriculum was failing to capitalize on the rich informal knowledge that students possess even before instruction. In particular, they say instruction often fails to take advantage of the natural problem-solving abilities of students when word problems are introduced.

Perkins and Simmons (1988) used the term *misunderstandings* to describe students' conceptions that may be different or in conflict with expert concepts. *Naïve concepts* are one subclass of misunderstandings and they characterize novice students. These concepts are usually conceived before formal instruction (just like informal/intuitive knowledge) and are usually strongly constructed to the extended they can be applied to multiple situations without failure. Naïve conceptions disregard counter-arguments and students who hold them tend to have a strong conviction, making it very difficult to change them. Naïve conceptions have been attributed in part to generalizations of real-life phenomena by students.

Jose (1989) and Resnick (1983) use the terms misconceptions and *naïve theories* to describe student conceptions that can interfere negatively with learning when students try to apply them in new contexts. According to Resnick students come to school with naïve theories that are constructed through everyday experiences. Students reluctantly let loose naïve theories because they were actively involved in creating them. As seen earlier, informal knowledge and intuitive knowledge are acquired in a similar way.

Instruction, Induction, and Student Conceptions

Students who have gone through considerable formal instruction hold ritual conceptions. Students with ritual conceptions often develop a high degree of procedural problem-solving skill in working with textbooks problems, but are not able to solve similar problems if they are presented in different contexts (Perkins & Simmons, 1988). In other words, students with ritual conceptions lack conceptual understanding although they possess a lot of procedural knowledge. Korner (2005) defined misconceptions as pieces of *wrong knowledge* that can arise as a result of insufficient knowledge or missing concepts. Korner's study demonstrated how instruction that emphasizes conceptual understanding can help reduce student misconceptions. The term *naïve conception* refers to the type of misconception characterized by an underdeveloped understanding of a concept or one that is based on insufficient information and not incorrect assumptions (Capraro, Kulm, & Capraro, 2005).

The phrases *mind bugs* and *flawed knowledge* refer to misconceptions that arise as a result of incomplete or misguided learning (VanLehn, 1990). The term "bug" has been used by some researchers (Ben-Zeev, 1995; Hennessy, 1993) to refer to slight and incorrect modifications to otherwise correct problem-solving procedures or algorithms. In other words, the term bug is used to refer to small, local misconceptions which

VanLehn also calls procedural misconceptions. Application of buggy algorithms results in what are called *error patterns* (Ashlock, 2006), or *rational/systematic errors* (Ben-Zeev, 1995; Hatano, Amaiwa, & Inagaki, 1996). Bugs arise from induction. In particular they arise as a result of students either over generalizing or over specializing concepts they have learned in the classroom (Ashlock, 2006; VanLehn, 1990).

Clements (1982) used the term *careless errors* to refer to errors whose origin was unaccounted for but were shown by students who were described as being weak arithmetically or had a poor grasp of mathematical language. These students would give correct answers to questions on one occasion, but supply wrong answers on other occasions. This characterization is similar to the one given to bugs.

Characterizations of Student Conceptions in Other Subject Areas

In other fields of study outside of mathematics, research on student misconceptions has also characterized student conceptions in more ways than one. For example in science and particularly in physics, student conceptions have been characterized as *preconceptions* (Clements, 1982; Glaser & Bassok, 1989) and McCloskey (1983) talked about *naïve theories* of motion developed on the basis of everyday experience and whose assumptions are consistent across individuals. Naïve theories are inconsistent with fundamental principles of physics (McCloskey). The concept of naïve theories is the same as that of naïve concepts seen earlier (Perkins & Simmons, 1988) in that both develop from real-life experiences. Other terms that have been used include *alternative conceptions*, *alternative beliefs*, *alternative frameworks*, and *pre-instruction conceptions* (Blosser, 1987). Alternative conceptions have been characterized as being persistent and being well embedded in an individual's cognitive structure. Just like intuitive knowledge, they are hard to teach away and are not easily detected by conventional methods.

Fractions, Decimals, and Percents

Fractions are among the most difficult topics that students have to learn in elementary school and children often find the topic "nonsensical and mysterious" (Siebert & Gaskin, 2006, p. 1). Rational number concepts are one of the most complex and most important mathematical ideas that are encountered by children before secondary school (Behr, Wachsmuth, Post, & Lesh, 1984).

According to the National Council of Teachers of Mathematics (2000), a solid understanding of different representations for fractions, decimals, and percents is required if one is to work flexibly with rational numbers. As students solve problems in context, they should consider advantages and disadvantages of various representations. Students should also be able to compare and translate among number forms to solve problems (NCTM, 2000).

Student Misconceptions: Fractions, Decimals, and Percents

A common misconception among students is the thinking that there is no relationship among fractions, decimals, and percents (Hackett, 2002; Pagni, 2004; Sweeney & Quinn, 2000). This misconception is reinforced by instruction that first focuses on computations with either fractions or decimals only (Pagni) or instruction that teaches the topics of fractions, decimals, and percents in isolation (Sweeney & Quinn). Students can be helped to develop a clearer understanding of representing a number as fraction or as a decimal by using the number line to show both names of the number. Showing students that fractions and decimals are merely different representations of the same number can be achieved by asking students to carry out an investigation in which they perform side-by-side computations that yield equivalent results (Pagni).

Many students lack a good understanding of percents. Responses given by college students to a question that asked them to find the percent that is the same as 1/3 included 33%, 0.03%, 33.3%, and .33%. Some gave 33 1/3 % as a solution, an indication that their understanding of percent was tied to the algorithm for converting a fraction to a percent usually taught in elementary school (Zambo, 2008). The algorithm can be described as follows: to convert a fraction to percent, one divides the numerator by the denominator to at least the hundredths place, moves the decimal two places to the right, and then adds a percent sign. Although the algorithm is effective, it does not provide students with an opportunity to understand the underlying meaning of percent. The meaning of percent is revealed in a different method of converting a fraction to a percent in which one finds the equivalent fraction with a denominator of 100. The hundreds grid can be used to represent percents and one way to do that is to let students discover that ¼ is equal to 25 percent by shading ¼ of the grid and counting the shaded squares, an approach that relies on understanding fractions as part of the whole. A lesson common, but possibly more beneficial approach to converting fractions to percents is based on the understanding of a fraction as a part of a group. With this understanding the fraction ¼ can be thought of as 1 out of a group of 4 and to represent ¼ and write it as a percent students would mark and count 1 out of every group of 4 squares on the hundreds grid.

Discussions and guidance should help students convert to percents fractions like 1/8 that do not result in whole number percents (Zambo, 2008).

A study by Jigyel and Afamasaga-Fuata'i (2007) revealed that students were more familiar with representing fractions with geometric figures, in particular circles than they were with representing fractions numerically. Although students could explain equivalent fractions when they were presented with figures, they had difficulties explaining the same fractions when presented numerically. In addition some students perceived the numerator and denominator of a fraction as two separate and unrelated numbers resulting in misconceptions when comparing area and numerical representations of equivalent fractions. Among the misconceptions that students exhibited was thinking that 4/6 is double 2/3. Students also limited the idea of fraction to the part-whole model (Jigyel & Afamasaga-Fuata'i). This falls short of students developing an understanding of fractions as parts of unit wholes, as parts of a collection, as locations on number lines, and as quotients or ratios as suggested by the Number and Operations Standard for grades 3-5 (NCTM, 2000). According to Thompson and Saldanha (2003) understanding "*a/b*" as a part-whole relationship, as many students do is unproblematic until students try to interpret numbers like 7/3 and 8(3/7).

In a study that investigated if students compared parts of a rectangle by reasoning if two fractions were equivalent or by making perceptual judgments (see Figure 1), Kamii and Clark (1995) found that 44% of fifth graders thought *a* and *c* were the same amount while 38% thought the triangular half was more, that c was more than a although the two rectangles had the same dimensions. Twenty-three percent (23%) argued that although both *a* and *c* were halves, the amount depended on how the rectangle was cut. Fifty-one percent of sixth graders said *a* and *c* were the same amount, while 17% persisted in thinking the triangular half was bigger. The researchers pointed to the existence of a conflict between operative knowledge which says ½ = ½ and figurative, perceptual knowledge which seemed to support the idea that the triangular half was bigger. Given figure X students were asked how many eighths equal 3/4. Only 32% of the 6^{th} graders gave the correct answer and explanation and 46% tried to solve the task spatially and figuratively, trying to fit the strips into the three-quarters area. Thirty-six percent of the 6^{th} graders rejected that ¾ = 6/8, while 49% of fifth graders did. Only 13% of fifth graders were able to independently reason that ¾ = 6/8. The researchers attributed these student difficulties to traditional instruction that teaches equivalent fractions perceptually and figuratively with pictures and manipulatives. Traditional instruction does not also give students an opportunity to think, struggle, and invent their own equivalent fractions, but teachers often tell students that certain fractions are equivalent. Traditional instruction was also criticized for teaching proper fractions first and then improper fractions, and not teaching them at the same time so students can think about parts and wholes at the same time.

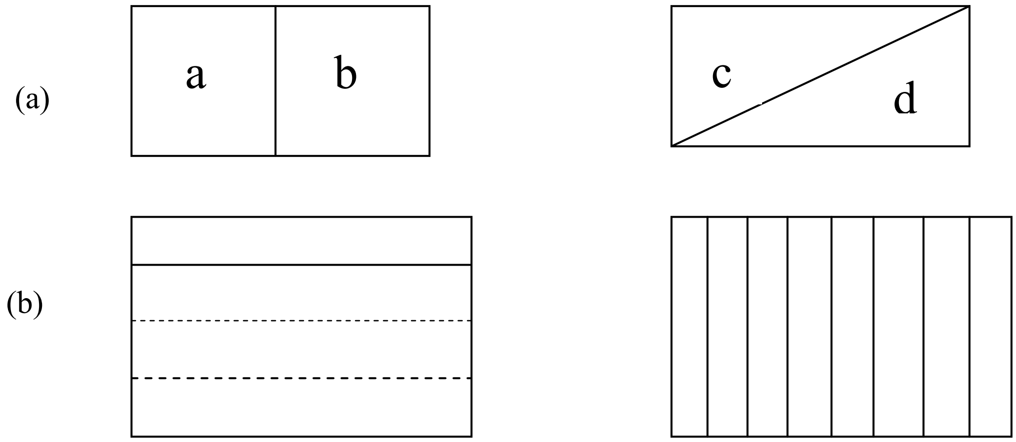


Figure 1: The ways in which paper was cut to test (a) 1/2 = 1/2 (b) 3/4 = 6/8.

Natural Language and Student Misconceptions

Teachers need to pay close attention to the language students use when they express their mathematical thinking because expressions used have underlying images. The use of the phrase “out of” or the term over as in “5 out of 8” and “5 over 8” for 5/8 are problematic in that students conceive being given 8 things and then taking 5 from those 8 things. They do not think of the 8 as eighths and the 5 as 5 one-eighths, but instead treat the numerator and denominator as merely whole numbers (Siebert & Gaskin, 2006).

Ashlock (2006) argued that teachers should not be in a hurry to have students use precise mathematical language. The author said

> When our students are learning mathematical vocabulary, they need to make connections with terms and concepts they already know. It is often helpful for them to connect with root meanings and related words. For example, the “nom”

> in denominator means “to name”. A denominator names the fraction; it indicated the kind of fraction, the size of the parts. And “numer” in numerator suggests “number.” A numerator tells the number of parts (Ashlock, p.70).

Mathematical Thinking and Misconceptions

A study by Herscovics and Linchevski (1994) revealed a cognitive gap between arithmetic and algebra in seventh graders. Students demonstrated difficulties of a pre-algebraic nature. When dealing with algebraic expressions, there was a tendency by students to detach a numeral from the preceding minus sign. Students also had problems accepting the equal symbol to denote decomposition into a difference as in $23 = 37 - n$, leading some students to read such equations from right to left.

According to Hatano (2003), mathematical knowledge in the form of laws or formulas can be transmitted to the learner. However before the laws can be used, the learner has to understand them through a reconstruction in the mind in which the new knowledge is interpreted in relation to prior knowledge. The existence of procedural bugs and misconceptions is evidence that students construct mathematical knowledge by themselves. Knowledge is not acquired by transmission alone, but when students are for example given a rule or algorithm they try to inductively construct something subjectively tenable (Hatano). Student misconceptions often arise when students either over-generalize or over-specialize new mathematical concepts they encounter in the classroom (Ashlock, 2006). Students tend to over-generalize meanings of symbolic representations for whole numbers to fractions and they also over-generalize meanings of symbolic representations for fractions to whole numbers (Mack, 1995). Students who lack a strong conceptual understanding of rational number concepts continue to have interference from their knowledge of whole numbers resulting in them treating the numerator and denominator of a fraction as separate numbers on which they can operate independently (Behr, Wachsmuth, Post, & Lesh, 1984). Over-generalizations and over-specializations by students are in part attributed to instruction that offers very limited examples. Teachers should present multiple examples and where possible also give non-examples of concepts that are being discussed to minimize the chances of students building misconceptions by induction from examples and prior knowledge (Ben-Zeev, 1995; Ben-Zeev & Star, 2001).

Instruction and Teacher Effects

Student academic achievement is mainly determined by the effectiveness of teachers. If students are assigned to consecutive ineffective teachers, their achievement can be heavily and negatively impacted in both the short and long terms (Rivers & Sanders, 2002; Sanders & Horn, 1998). According to the NCTM (2000, p. 15-16),

> In planning individual lessons, teachers should strive to organize the mathematics so that fundamental ideas form an integrated whole. *Big ideas encountered in a variety of contexts should be established carefully*, *with important elements such as terminology, definitions, notation, concepts,* and skills emerging in the process. …Students learn mathematics through the experiences that teachers provide. Thus, students' understanding of mathematics, their ability to use it to solve problems, and their confidence in, and disposition toward, mathematics are *all shaped by the teaching* they encounter in school [emphasis added].

Lessons on fractions can be made more motivating and successful if teachers show students how fractions apply to their lives. Using manipulatives can make lessons more active while at the same time giving teachers more opportunities to observe what students are doing and how they are doing it. Having open discussions can also help teachers identify any misconceptions that students might have, such as thinking that multiplying a fraction by a multiplicative identity 1 other than 1/1 yields a larger fraction (Naiser, Wright, & Capraro, 2004).

Missed Opportunities and Limited Experiences

Operating with rational numbers and integers are the two intermediate skill areas where students entering high school are most in need of extra-help. These two domains are conceptually challenging, procedurally complex, and vital to success in standards-based high school mathematics. They cannot be mastered by simply extending one's knowledge of whole number operations. They are made more challenging by the fact that implicit rules learned for operating with positive whole numbers do not apply (Kilpatrick, Swafford, & Findell, 2001; Stavy & Tirosh, 2000). Although knowledge of whole number operations is a primary focus of instruction in upper elementary and middle school, not all middle school students receive sufficient and effective instruction on the topic (Balfanz, McPartland, & Shaw, 2002).

Instruction that emphasizes a relational definition can have an impact on how students understand the equal sign (Baroody & Ginsburg, 1983). In a yearlong study that examined interpretations of equality and the equal sign by third graders, Sáenz-Ludlow and Walgamuth (1998) found students were able to expand their conceptualizations of the equal sign through active participation in discussions, appropriate mathematics tasks, and the teacher's intellectual sensitivity to the balance between teaching and learning. Students had initially interpreted the equal sign as an operator symbol, but by the end of the school year they had developed a relational understanding of the equal sign. Adequate instruction over an extended period of time can help students develop a level of thinking sufficient to deal with questions of the order and equivalence of fractions (Behr, Wachsmuth, Post, & Lesh, 1984).

An understanding of the concepts of equality and variable influences success in algebra, in particular it influences success in solving problems, the strategies used in solution processes, and the justifications provided for the solutions. Many middle school students lack a sophisticated understanding of the equal sign while even fewer hold a relational view of the equal sign. A lot of them understand the equal sign as an operational symbol. American mathematics lessons (K-12) rarely focus explicitly on the equal sign and its meaning. These limited experiences may explain in part why students in middle grades tend to interpret the equal sign as an operational and not a relational symbol (Knuth, Stephens, McNeil, & Alibali, 2006; McNeil & Alibali, 2005).

Understanding of the equal sign as an operational symbol persists throughout elementary school and into middle school where very few middle school students hold a relational view of the equal sign. Some high school students demonstrate a lack of understanding of the equal sign through the errors they make when solving equations (Kieran, 1981).

Teacher Knowledge

“To assume that the content of first-grade mathematics is something any adult understands is to doom school mathematics to a continuation of the dull, rule-based curriculum that is so widely criticized” (Ball, 1988, p. 23). The National Council of Teachers of Mathematics (2000) says “teachers need several different kinds of mathematical knowledge—knowledge about the whole domain; deep, flexible knowledge about curriculum goals and about the important ideas that are central to their grade level” (p. 17). According to Ball, Hill, and Bass (2005), the quality of mathematics teaching depends on teachers’ content knowledge and many U.S. teachers lack sound mathematical understanding and skill. To gain a strong conceptual understanding and understand connections among topics, teachers should revisit the mathematics that they teach (Mewborn, 2003).

Necessary Teacher Knowledge

The mathematical knowledge that teachers need to know can generally be described as (a) topics and ideas fundamental to the school curriculum (b) tools and skills for reasoning about mathematical claims, ideas, representations, and solutions; and sensibility about what constitutes adequate proof (c) fluency and care with mathematical language and notation and (d) familiarity with applications of mathematics (Ball, 2003). Mathematical knowledge for teaching has been characterized as being either subject matter knowledge (SMK) or pedagogical content knowledge (PCK). Common content knowledge (CCK), specialized content knowledge (SCK), and knowledge at the mathematical horizon all fall under SMK. Knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of curriculum fall under PCK (Ball & Sleep, 2007).

Deficiencies in Content Knowledge

A study by Post, Harel, Behr, and Lesh (1991) in which middle school teachers were tested on the concepts of rational number concepts: part-whole, decimals, ratios and percents, proportionality, multiplication and division, revealed that many teachers did not know enough mathematics. Only a few of those teachers, who were able to solve the problems correctly, were able to explain their solutions in a satisfactory manner. The mean of acceptable explanations for the 44.7% of the teachers who were able to compute the correct results was 20.3% and 27% respectively for two versions of the tests. The authors say some of the misunderstandings they had found in students were also present in teachers.

A study by Southwell and Penglase (2005) revealed that primary pre-service teachers had weaknesses in understanding concepts about place value, operations with common fractions, multiplication of decimal fractions, percentages and measurement. Khoury and Zazkis (1994) examined reasoning strategies and arguments given by pre-service school teachers as they solved problems on fractions with different representations. They were asked to compare between two different fractions having the same numerical representation and to compare between different notational representations of the same fraction. The majority of them believed fractions change their numerical value under different symbolic representations.

A study involving pre-service primary school teachers revealed weaknesses in recognizing patterns and relationships. There was also a correlation between insecure subject knowledge and poor planning and teaching (Goulding, Rowland, & Barber, 2002). In another study, the majority of participants were unable to provide word stories for division by a fraction and subtracting a negative number. This demonstrated a limited understanding of the operations of division and subtraction (Goulding & Suggate, 2001).

Subject matter preparation of teachers is rarely a focus of teacher education. Unfortunately teaching itself does not produce the kind of understanding that teachers need to teach. Learning about the understandings of mathematics that prospective teachers bring with them to teacher education can help universities to work with prospective teachers so that they move toward the kinds of mathematical understanding needed to teach mathematics well (Ball, 1988).

Thompson and Thompson’s studies (1994, 1996) concern middle school teachers involved in a teaching experiment in which they taught concepts of rate to one student. Although the teachers’ conceptualizations of rate were strong, they had difficulties speaking conceptually about rate to the student. By explaining the concepts mainly in terms of whole numbers and whole number operations, the teacher reinforced the student’s incorrect additive instead of multiplicative thinking. The teachers’ difficulties appeared to be rooted in their language (as a representational medium).

Teachers’ knowledge of mathematics can impact instructional practices. A study involving an experienced fifth grade teacher (Stein, Baxter, & Leinhardt, 1990) revealed that the teacher's knowledge of functions and graphing was missing several key mathematical ideas. In addition, the teacher's knowledge was not organized so as to provide easily accessible, cross representational understanding of the domain. The impact was a lack of provision of groundwork for future learning, overemphasis of limited truths, and missed opportunities for promoting the development of meaningful connections between key concepts and representations.

Inattention to mathematics content and lack of subject matter knowledge on the part of the teacher can result in students inappropriately applying mathematical procedures. This can be more devastating if students do not question the reasonableness of their solutions (Heaton, 1992). In another study in which the teacher was teaching the concept of mean, emphasis was on the steps of the procedure for finding the mean and lack of reflection on the reasonableness of solutions leading both teacher and students to unknowingly and incorrectly compute means (Putnam, 1992).

Ding (2007) found that teachers who had a conceptual understanding of equivalent fractions and basic mathematical ideas were able to teach for understanding. The use of inaccurate verbal representations by some of the teachers in the study led to student errors with the most common error being what the researcher calls the doubling error. Students who made the error wrote expressions like $\frac{3}{4} \times 2 = 6/8$ instead of $\frac{3}{4} \times 2/2 = 6/8$. Some were writing $\frac{3}{4} + \frac{3}{4} = 6/8$ because they had seen expressions such as $(3 + 3)/ (4 + 4) = 6/8$.

A teacher's flexible understanding of fractions can help them to both adjust instruction to accommodate mathematical ideas that are not necessarily at the forefront of their thinking and to emphasize connections among mathematical representations when curriculum materials supporting such emphasis are available (Wilson, 1994).

Textbook and Representation Effects

Many mathematics teachers use the textbook as a guide to implement the curriculum. A review of middle school mathematics textbooks revealed that a few excellent middle grades mathematics textbooks exist, and none of the best rated are popular commercial textbooks. Most of the textbooks are inconsistent and weak in coverage of conceptual benchmarks in mathematics. In addition, a majority of the textbooks are weak in their instructional support for both students and teachers. The evaluations also revealed that many textbooks provide little development of mathematics idea in middle school grades and a majority of them do not take into account student ideas or promote student thinking (Kulm, Roseman, & Treistman, 1999).

In a comparative study involving 2^{nd} and 6^{th} graders from China and the U S (Ding, Li, Capraro, & Capraro, 2007) and in a study involving middle school students (McNeil et al., 2006), the researchers attributed an operational understanding of the equal sign by students to the fact that textbooks read by both teachers and students rarely presented the equal sign in contexts that called for a relational interpretation.

Naiser, Wright, and Capraro (2004) observed that in classrooms where students did seat work, worksheets, or watched the teacher as they solved problems on an overhead as opposed to using manipulatives; the students were not actively involved in the lesson and were often engaged in other things than their lesson. Grids, pattern blocks, fraction strips, paper folding, tiles, and base ten blocks were among the manipulatives that were used by teachers in fraction lessons. Using real world problems and building on students' prior knowledge were two of several strategies that teachers used to engage students in fraction lessons (Naiser, Wright, & Capraro).

Representations can be considered as the language of mathematics (Coulombe & Berenson, 2001). The importance of representations lies in the fact that representations are vehicles for learning and communicating. Because representations come in different forms, students can use combinations of representations to gain more information than would be possible with a single representation (Friedlander & Tabach, 2001).

Students' academic achievement and learning can be affected by differences in external representations due to the fact that some representations are easier to comprehend than others and some representations elicit more reliable and meaningful solution strategies than others when considering decimals, fractions, and percents. In particular, students have difficulties comprehending formal symbolic representations of quantitative relations (Koedinger & Nathan, 2004). Comprehensibility of representations can influence student achievement while use of multiple representations can help enhance student learning (Sun, 2005). Because students can touch, move, and often stack them, manipulatives used appropriately can provide children with opportunities to compare relative sizes of objects representing mathematical ideas such as fractions. Manipulatives also give children an opportunity to identify patterns and put together representations of numbers in multiple ways (Clement, 2004). A representation can be effective in one situation and also have the potential of being over generalized or misapplied in other situations. Understandability, clarity, transferability, and flexibility are important aspects of representations in that they impact how students transition from using concrete tools to performing operations with numbers and symbols (Capraro & Capraro, 2006).

Teachers' choice of instructional representations can be greatly influenced by textbooks (Sun, 2005). Representations used by teachers influence the representations their students use. This in turn impacts on problem solving (Cai & Lester, 2005). The study by Cai and Lester revealed an overwhelming use of symbolic representations for solutions by Chinese teachers, whereas U.S. teachers relied heavily on verbal explanations and pictorial representations.

According to Preston and Garner (2003), in addition to helping students visualize mathematical situations, pictorial representations are a comfortable approach for a majority of middle grades students. The drawbacks include the facts that students often make assumptions that go beyond the statement of the problem, such as a triangle drawn as equilateral, and some students have poor drawing skills. Natural language, which falls in the class of verbal representations, often helps students connect problems to the real world. Unlike precise mathematical language, natural language has the disadvantage that that it can be ambiguous. Algebraic representations have the advantage that they provide a concise and general statement of a situation that can be manipulated more easily once they are created. Algebraic representations may fail to communicate meaning to other students and they are initially the most difficult for most students due to a lack of previous experience (Preston & Garner).

The following two scenarios (Clement, 2004, p. 97) demonstrate that students may find it easier to comprehend real-life situations than it is to comprehend written symbols and that the connections that children make between language and written symbols may also differ from the connections made by adults. Written symbols are often more abstract to students than other representations (Clement).

Scenario 1

Teacher: Can you solve this problem? (*Gives student paper with 4 – 1/8 written at the top*)

Student: (*Writes 3/8 as answer*) Three-eighths. I subtracted 1 from 4, and then kept the denominator, eight, the same.

Teacher: Suppose you had four large brownies and you ate one eighth of one brownie. How many brownies would you have left?

Student: (*Pauses, then draws four rectangles, partitions one of the rectangles into eight pieces, and shades one of the pieces*) Three and seven eighths. (*Writes 3 7/8*)

Scenario 2

Teacher: I will say a number, and you write it. Please write for me the number *one half*.

Student: (*Writes 1 1/2*)

Teacher: Please write for me the number *one and one-half*.

Student: (*Writes 11/2*) It is the same thing.

Teacher: (*Writes ½ and points to it*) What would you call this?

Student: Half

The ability to use representations is essential to learning mathematics. Ability to generate and use representations should be assessed in students who struggle in mathematics and such assessment should determine the relevancy, quality, and completeness of the representation. Assessment should not only focus on an end product, such as a graphical representation, but also evaluate when, where, and, how a student uses a representation as this may be more beneficial (Scheuermann & van Garderen, 2008). Students are expected to develop meaning for symbolic representations as part of their middle school mathematics experience. Because most of the time students lack a deep understanding of algebraic symbols that they write, teachers must move beyond the focus on manipulating symbols to include a focus on the internal meaning ascribed by students to written symbols (Lannin, Townsend, Armer, Green, & Schneider, 2008).

Teacher Credentials

A teacher's experience, test scores on licensure, and regular licensure all have positive effects on student achievement particularly in mathematics (Clotfelter, Ladd, & Vigdor, 2006, 2007). A study involving 12th grade teachers (Goldhaber & Brewer, 2000) also revealed that mathematics teachers with a standard certification had a positive and statistically significant effect on students' test scores relative to their counterparts with either a private school certification or no certification in the subject area. In a review of studies on teacher credentials (Wayne & Youngs, 2003), the researchers concluded that high school students learned more mathematics from teachers who had taken more coursework in mathematics or had additional degrees.

METHODS

Research Design

Mixed methods research is formally defined as the "class of research where the researcher mixes or combines quantitative and qualitative research techniques, methods, approaches, concepts or language into a single study" (Johnson & Onwuegbuzie, 2004, p. 17). According to these authors, the first three steps of an eight step mixed methods research process model are (1) determining the research question; (2) determining whether a mixed design is appropriate; and (3) selecting the mixed method or mixed-model research design. "Most good researchers prefer addressing their research questions with any methodological tool available" and method is secondary to research questions (Tashakkori & Teddlie, 1998, p. 21). Because some of the research questions in this study required qualitative techniques and some required quantitatizing of qualitative data to address them, I employed mixed methods research techniques.

According to Onwuegbuzie and Leech (2005), mixed methods research designs can be classified according to (a) level of mixing (partially mixed vs. fully mixed); (b) time orientation (concurrent vs. sequential), and (c) emphasis of approaches (equal status vs. dominant status). The research design for this study which is illustrated in Figure 2 can best be described as being partially mixed and sequential with the qualitative approach enjoying a dominant status. I used qualitative techniques in the first phase of the investigation and some of the data from this qualitative phase was studied quantitatively in the second phase of the study. Figure 1 gives a visual illustration of the research design and also contains information on the different types of data studied as well as information on when and how these data were collected. Appendix A gives a description of the types of data that were used in this study. The same appendix also summarizes the origins of these data and how they were collected.

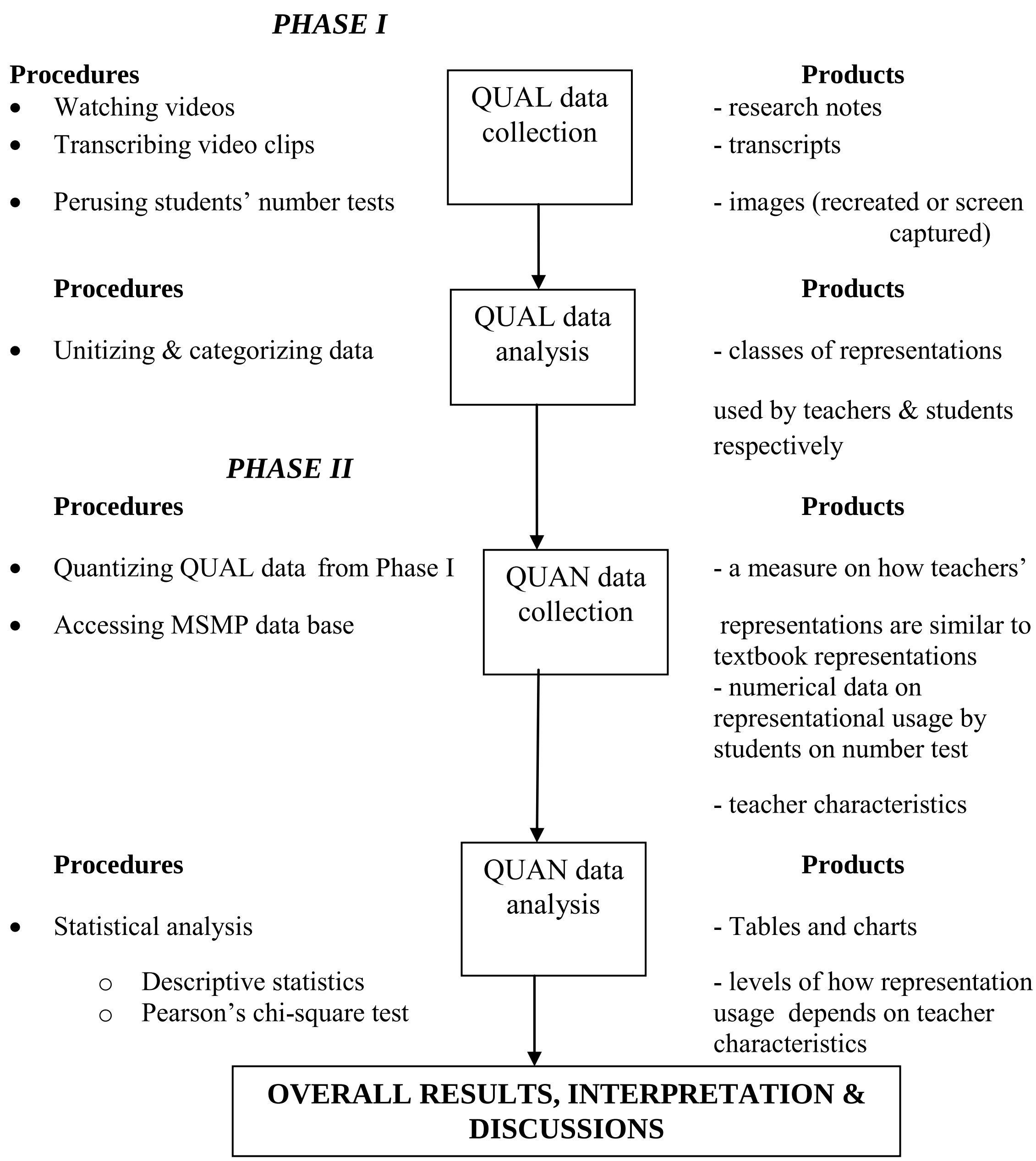


Figure 2: Research design.

Variables

The variables that were studied include types of representations used by both teachers and students (QUAL), and the quantitative variables which are, (a) *x,* a measure of how teachers' representations are similar to textbook representations as represented by the codes of representations identified in question 1, (b) curriculum, in particular the textbook that the teachers were using, (c) highest degree obtained by each teacher, (d) certifications, (e) number of years spent teaching mathematics, (f) number of years teaching mathematics at grade level, (g) number of hours completed on professional development (PD) related to their textbook, (h) the total number of days spent on the Interagency Education Research Initiative (IERI) professional development between 2002 and 2005, and (i) numerical data on representational usage by students on a number test that will be discussed in the instruments section. Variables (b)-(h) which were obtained from the Middle School Mathematics Project (MSMP) database in teacher characteristics and students performance were the used together with (a) to address the question of how teachers' representational usage was dependent on teacher characteristics. The variables (e)-(h) were collected through a survey as part of the Interagency Education Research Initiative (IERI) professional development (DeBoer et al., 2004). The variable (i) was useful in studying how students' representation choices on the number test were influenced by teacher selected representations.

I thought it would be important to investigate how the independent variables discussed in the last paragraph were related to representational usage by teachers in this study due to the fact that using representations effectively was one of the three instructional criteria that were the initial focus of the IERI professional development. A decision to focus on (a) using representations effectively, (b) probing student understanding, and (c) guiding student interpretation and reasoning was reached after it was observed by researchers that the teachers' instruction was not focused on building a conceptual understanding of the learning goals for the students (DeBoer et al., 2004).

Participants

The sixteen (16) teacher participants for this study were purposively selected from the set of all teachers who participated in the MSMP which started in 2002 and ended in 2007. The project involved middle school mathematics teachers from Delaware and Texas. I used a purposive sampling technique known as criterion sampling. To use this technique, a researcher sets up a criterion or set of criteria and then identifies cases that meet the specifications (Mertens, 2005; Onwuegbuzie & Collins, 2005). Teacher participants were chosen if they met the following two criteria: (a) participated at least once in the Texas MSMP and (b) taught at least one lesson on converting among fractions, decimals and percents at the 6th grade level.

Since some of the teachers who met the above criteria participated in more than one year of the project, only the videos recorded in the year in which they last participated in the project were used in this study. The same sample of teachers (and videos) was used for both the qualitative and quantitative phases of the study. All students whose teachers were chosen to participate in this study constituted the student participants. Table 1 shows the number of students taught by each teacher during the school year in which selected videos for this study were recorded. A total of five hundred eighty one (N = 581) students were part of this study.

Table 1

Number of Students Taught by Each Teacher

Teacher	T1	T2	T3	T4	T5	T6	T7	T8	T9	T10	T11	T12	T13	T14	T15	T16
Number of students	73	50	17	23	37	71	43	26	42	10	65	17	22	50	15	20

Teacher Characteristics

The three textbooks used by teachers in this study were *Middle Grades Math Thematics* (Bilstein, et al., 1999), *Mathematics Applications and Connections* (Collins et al., 1998), and C*onnected Mathematics* (Lappan, Fey, Fitzgerald, Friel, & Phillips, 1998). Table 2 shows how many teachers were using each of these three textbooks.

Table 2

Number of Teachers Using Each of the Three Textbooks

Textbook	Number of teachers
Middle Grades Math Thematics	7
Mathematics Applications and Connections	2
Connected Mathematics	7

The highest degrees attained by teachers in this study were either a bachelors or a masters' degree. The three different certifications were at the elementary, middle school, and high school levels. The number of years teachers spent teaching mathematics ranged between zero and twenty-five years. Nine of the sixteen teachers had spent between zero and five years teaching mathematics at the sixth grade level, while the maximum number of years spent teaching mathematics at the sixth grade fell in the range of 16-20 years. In addition to having spent hours ranging from zero to over 200 on professional developments about mathematics and education in general, eight of the teachers had professional development on the textbook they were using. Seven of these eight teachers had between 1 and 50 hours of professional development on the textbook while the eighth teacher had between 51 and 100 hours.

The IERI professional development sessions were conducted from 2002 through 2004 (DeBoer et al., 2004). Table 3 summarizes the data on the total number of days of the professional development attended by the sixteen teachers who were part of this study.

Table 3

IERI Professional Development Data

Variable	Min	Max	Median	Mean	Std. Deviation
Total number of days attended	1	24	7.75	9.16	6.21

Instruments

Two instruments were used for data collection in this study. The first instrument (see Appendix B) was used to determine the extent of similarity between teachers' representations as seen on the videos with textbook representations. The numbers 0, 1, and 2 were used to code the teachers' representations. The number 0 was used to code a teacher's representation totally different from and in a different class from the representation used or suggested in the textbook or a teacher's representation not used in the textbook at all. Examples would be when a teacher used a figure to solve a problem when the textbook suggested using manipulatives or when a teacher used the word top to refer to the numerator of a fraction. The code 1 meant the teacher's representation and the textbook representation were similar (in the same class), but the teacher had modified their representation, which may or may not have resulted in the two representations conveying different meanings as a result of the teacher's representation being an idiosyncratic representation. An example would be when instead of writing an equation; a teacher used the forward arrow instead of the equal sign symbol. The code 2 was used for teachers' representations that were in the same class and exactly or almost the same as the textbook representations. Examples would be when a teacher copied a representation given in the textbook and displayed it on an overhead projector or when a teacher used a rectangular geometric figure to represent a fraction instead of a circular figure as the textbook. In instances where a teacher simultaneously used verbal and visual representations, such as speaking as they wrote on the board, the verbal and visual representations were coded separately. To obtain an estimate of coding reliability, a knowledgeable other was trained who recoded some of the videos used in this study. Percent agreement will be reported as an estimate of reliability.

A second instrument was a number test that was designed to measure students' understanding of the learning goal (DeBoer et al., 2004). The learning goal was that students were able to "use, interpret, and compare numbers in several equivalent forms such as integers, fractions, and decimals" (American Association for the Advancement of Science, 1993). Although pre and post number tests were administered to all the students in this study, I only focused on the posttest because the goal was to study how teachers' representational usage influenced representational choices by students. The posttests consisted of 16 items; nine multiple choice questions, six short response questions, and one extended response item composed of five parts.

Qualitative Phase

Data Collection

All selected videos were watched with close attention paid to representations (verbal or visual) of the numbers and mathematical procedures or algorithms used by the teachers and students when they were discussing converting among fractions, decimals, and percents. For purposes of this study, a "chunk" was defined as a period of time during which a teacher or student used or talked about a representation deemed necessary for addressing any of the research questions. Each selected video was watched and divided into chucks. It is important to point out that I was not necessarily concerned with the lengths of the chunks. I was mostly concerned with the types of representation used and where that information could be found on the video tape. Careful research notes such as a record of time stamps, type of representation used and whether it originated from the teacher or student, how the representation differed from the representation used or suggested in the textbook and whether representations were idiosyncratic or not were noted. Any relevant information was kept in a spreadsheet (see Appendix B). Some of the selected video clips were either transcribed and/or screen captured or re-created as needed, depending on whether a teacher or student used a verbal, visual or a combination of both verbal and visual representations. All selected videos were watched before the qualitative analyses explained in the next section began.

In addition to observing the representations that students used in the classroom, I also studied students' responses on the number test. I studied the different representations that each student used on the number test with particular attention paid to students' use of idiosyncratic representations. The main goals of studying students' responses were to see if any of these responses revealed misconceptions and if students' representational forms corresponded to various degrees of correct solutions. Careful research notes on representational usage by each student were kept in a spreadsheet.

Analyses

Qualitative data processing activities can involve unitizing and categorizing the data (Lincoln & Guba, 1985). According to Lincoln and Guba, a unit should have two characteristics;

> First it should be heuristic, that is, aimed at some understanding or some action that the inquirer needs to have or to take. Unless it is heuristic it is useless, however intrinsically interesting. Second, it must be the smallest piece of

> information about something that can stand by itself, that is, it must interpretable in the absence of any additional information other than a broad understanding of the context in which the inquiry is carried out (p. 345).

For the purposes of this study, a unit was defined to be all the information found within a chunk as defined above. This information included type of representation, whether it was idiosyncratic or not and the notes describing the extent of the similarity of the representation with the representation used or suggested in the textbook. In the categorizing stage of the analysis, I used a constant comparison method (Lincoln & Guba). The first step in the analysis process was to print all the research notes that I wrote while I was watching the videos. After sorting the last names of the teachers in alphabetical order, the teachers were renamed T1 to teacher T16 in an effort to conceal the teachers' identities. I then created a file with seventeen columns, the first column containing types of representations used by the teachers and the other sixteen columns labeled T1 to T16, one column for each teacher containing information on how the teacher used the representations. Starting with teacher T1, I put into categories (which are represented by rows in the spread sheet) the units according to the types of representations identified within each unit. The context in which each representation was used and how frequently it was used was also recorded. I also took note of the page, where in the printed spreadsheets, the time stamp of the chunk containing the unit can be found. With subsequent teachers, if they used a representation that had already been identified as being used by the previous teacher, I noted what context the representation was used in and where the chunk was located. If teacher T(i+1) used a representation that teacher Ti did not use, a new row describing the representation was created. The procedure was repeated until all units from the sixteen teachers were categorized. In addition to listing the representations that the teachers used, separate rows were also created to take note of instances where teachers used idiosyncratic representations, to note whether or not they emphasized using the multiplicative identity to obtain equivalent fractions, to note whether or not the teachers utilized opportunities to explicitly use the equal sign, and whether or not the teachers pointed to the limitations of some of the representations they were using.

The first step in analyzing the research notes on students' representation usage on the number test was to separate the research printed notes into sixteen groups according to who each student's teacher was. I then formed categories of the representations that students used by employing a constant comparison method (Lincoln & Guba, 1985) within each of the sixteen groups. The resulting representation categories for each group were then compared (to identify similarities and differences) with the categories of representations used by the group's teacher.

Quantitative Phase

Obtaining the Data

Two sets of numerical data for the quantitative phase were obtained by *quantitizing* the qualitative data obtained in the first phase of the study. Quantitizing data is a process by which qualitative data are treated with quantitative techniques to transform them into quantitative data (Sandelowski, 2000). The first set of data was obtained by coding qualitative data on representational usage by teachers using an instrument design specifically for this task (see Appendix B). I coded the qualitative data by comparing teachers' representations with representations used or suggested in the textbooks from which the lessons were adopted. The second set of data was obtained by counting frequencies of representational usage on the number posttest by each student.

Additional data about the teachers such as the number of years they had taught mathematics, the number of years they had taught mathematics at 6th grade, number of hours of professional development they had completed related to their textbook, to mathematics, and to topics of a more general nature, highest teaching qualification, and certification type were obtained from the MSMP database in teacher characteristics and student performance. These data were then analyzed using quantitative techniques as described in the next section.

Analyses

In addition to using descriptive statistics to characterize use of representations by both teachers and students, some of the quantitative data were analyzed using *Pearson's chi square test* to determine if frequency of use of certain representations by teachers was dependent on factors like the curriculum, highest degree obtained by teacher, certifications, number of years spent teaching mathematics, number of years teaching mathematics at grade level, number of hours completed on professional development related to their textbook, and the total number of days spent on the IERI professional development. The Pearson's chi square test was a chosen because of the small sample size (n = 16) which meant I could not use parametric tests which would have required, among other things, that the data be from a normally distributed population, an assumption that cannot be made with a sample size of 16. I also used tests of null hypothesis to investigate how the variables were related to each other. In order to carry-out chi square tests and hypotheses testing, some of the data sets were reorganized as indicated in Table 4. The first column gives the variables that were being investigated against extent of similarity between teachers' and textbook representations while the second and third columns give the values assumed by the original data and the reorganized data respectively. The extent of similarity between teachers' and textbook representations assumed values 0, 1, or 2 in the original data set. In instances where these data were reorganized, the new values were (0 or 1) and 2. When I was investigating how certification was related to representations usage, the levels of certification were narrowed from three to two by comparing teachers certified for middle school with teachers that were not certified for middle school. In investigating how representation usage was related to the total number of days that a teacher spent in the IERI professional development, I divided the group of sixteen teachers into two groups depending on whether the total number of days spent on the professional development was less than or greater than the median. The same two groups could have been obtained using the mean instead of the median.

Table 4

Data Reorganization for Purposes of Carrying-out Statistical Analyses

Variable	Original	Reorganized	New extent of similarity values
Highest degree	B, M	B, M	0 or 1, 2
Certification	MS, Elem, Math	MS, Other	0 or 1, 2
NYTMATH	0-5, 6-10, 11-15, 16-20 and 21-25	0-5, 6-10, 11-15, 16-20 and 21-25	0 or 1, 2
NYTGL	0-5, 6-10, 11-15, 16-20 and 21-25	0-5, 6-10, 11-15, 16-20 and 21-25	0 or 1, 2
TOTALNDAYS0205	1, 2, 3, …, 24	LTM and GTM	0,1, 2
NHPDTXT	0-50, 51-100	0-50, 51-100	0 or 1, 2

Note. B = Bachelors, M = Masters, MS = middle school, Elem = elementary, NYTMATH = number of years teaching mathematic, NYTGL = number of years teaching at grade level, TOTALDAYS0205 = total number of days spent on IERI professional development, LTM = less than median, GTM = greater than median, NHPDTXT = number of hours spent on professional development related to textbook.

RESULTS

This section is divided into sections according to the research questions. The first section is concerned with addressing the first research question whose goal was to identify the different types or classes of representations that teachers in this study were using. In the second section I focus on the use of idiosyncratic representations by teachers and how these influenced students' representational choices on the number test. The second section also discusses how teacher selected representations were similar to textbook representations. The third section deals with the question of how teachers' representation usage varied according to teacher characteristics. The section concludes with a discussion on how teacher selected representations are connected to student misconceptions as revealed in the classroom videos or on the number test.

Research Question 1

This section is concerned with studying the different types of representations used by the teachers who were part of this study. Data used in this section was obtained by watching classroom videos. To provide detailed accounts of the different representations that the teachers were using, the section is divided into subsections in which I will focus on verbal and visual representations separately. The question being addressed was stated as follows:

What types of external representations (verbal or visual) of fractions and mathematical procedures or algorithms do middle school teachers use in lessons on converting among fractions, decimals, and percents?

Verbal Representations and Characterizations of Fractions

The majority of teachers in this study were not observed giving a formal definition of a fraction. Two of the sixteen teachers (T2 & T14) defined a fraction as *part of a whole*. The following is a transcript of an episode in which T2 was talking about the definition of a fraction.

T: Okay. Number one: Definition of a fraction. Definition of a fraction. Number twenty-one (C*hooses a number out of a jar*). Do we have twenty-one?

S: Parts of a whole.

T: Parts of a whole. Excellent. Remember we talked about the candy bar or anything. We talked about a candy bar, and I'm only looking at a fraction of that candy bar. I'm looking at part of the whole candy bar. Or a pizza. If I am only looking at a fraction of a pizza, then I'm only looking at a part of the whole pizza. Okay. Excellent. Part of a whole. Okay.

In the process of defining a fraction the teacher uses real-world examples (candy bar & pizza). Defining a fraction as part of a whole is aligned with defining the denominator of a fraction as the *total number of pieces* by teachers T1 and T12. In teacher T12's class, the numerator was characterized as "the number of pieces of the whole" that are being considered. What follows is part of a discussion that was held in teacher T1's class. The discussion has to do with covering the star in Figure 3 first with triangles and then with rhombi. It took 12 and 6 triangles and rhombi respectively to cover the star and the goal of the whole exercise was to demonstrate that 12/24 = 6/12.

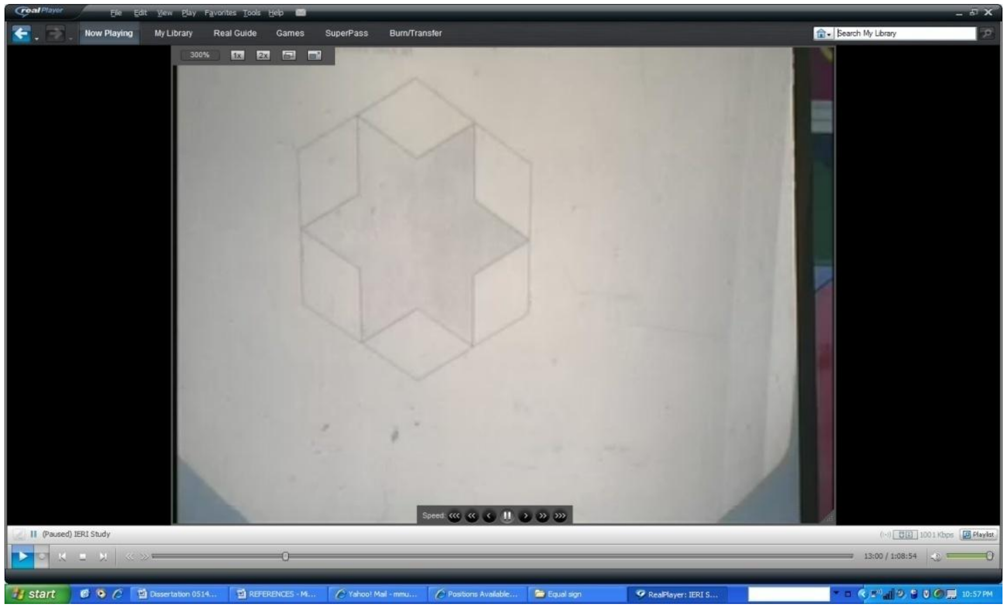


Figure 3: Showing 12/24 = 6/12 = ½ by covering part of a hexagon with pattern blocks.

T: What's the only thing that changed when we were … putting on our triangles and our rhombi? What's the only thing that changed? (*Calls student's name*)

S: (*Inaudible)* Our denominator

T: The denominator right. It's the only thing that changed was the denominator which tells, tells what? What does our denominator tell us?

S: *(Inaudible)*

T: How many? What does our denominator say? (*Calls upon student*)

S: That there is more or less triangles or rhombi …

T: You are going ahead of me …in a fraction what does a denominator tell? (*Calls upon student*)

S: (*Inaudible*) The total.

T: It tells us, right, the total number of pieces we have right. So when we were dealing with the rhombi we had twelve, right and then when we were dealing with the triangles we had … twenty-four but when it came to covering-up our star we had six out of twelve (*Writes 6/12*) with the rhombus and then what?

S: Twelve out of twenty-four.

T: Twelve out of twenty-four (*Writes 12/24*) with our triangles and both these equaled one-half … so that means those were equivalent. The only thing that changed was our denominator, but we could still say that it represented one-half

In the above discussion the teacher tried to explain what the denominator of a fraction means, but appeared to run into problems because the scenario they were talking about was highly contextualized. The teacher started out very well when she asked the question, what does our denominator tell us? In the scenario quoted above, the discussion of what a denominator was seemed to have shifted from the context they were initially discussing to a discussion of what the denominator in any fraction meant. The assertion by the teacher that it is only the denominators that changed from 12 to 24 when they shifted from talking about 6/12 to talking about 12/24 was also confusing in the sense that the numerators of the fractions also changed from 6 to 12.

The following discussion was carried out in T2's class about the fraction 2/4. The numerator 2 was referred to as the part and while the denominator was called the whole.

T: Two parts, right. First question on the quiz: What is a fraction? Parts of a whole. So in this case, the two is the part. Well, what would the four be? (*Calls upon student*)

S: The whole.

T: The whole. Right. Two would be the part. Four would be the whole. Now, go ahead and copy this down in there.

Although the definitions of a fraction (part of a whole), numerator (part), and denominator (total number of pieces or whole) given by teachers T1, T2, T12, and T14 as seen above were correct in the context they were used, that is, in the context where a teacher introduced the concept of a fraction by starting with a single figure or object which was then subdivided into equal parts, the definitions did not make sense and would be very confusing to apply when students encountered numbers such as 7/2 or say 7/ (3/5). The definition of a fraction discussed in teacher T1's class implied you cannot simply refer to an improper fraction as a fraction. These examples highlighted the misconceptions that can arise from developing definitions in highly contextualized situations. The same examples also demonstrated how complicated transitioning from the concrete (in this case pattern blocks and Hershey bars) to the abstract can be. The examples also demonstrated that assertions that are true in the concrete might not necessarily be true in the abstract.

Natural Language

In this section I discuss how some teachers used non-mathematical terms to characterize fractions or numerators and denominators. Teachers who used natural language were relating concepts that were being learned to contexts that students were mostly familiar with.

The words *top* and *bottom* were used by nine (9) of the sixteen (16) teachers at least once in place of the words numerator and denominator respectively. One context in which the words were used was when teachers T3 and T5 defined an improper fraction as a number in which the top is larger than the bottom. Not all teachers used, or permitted students to use the words top and bottom. Teacher T1 asked a student who used the words to explain what the words meant, guiding the student so they would use the words numerator and denominator instead. The following is an example of teacher T7 using the words top and bottom to refer to the numerator and denominator of a fraction.

> T: So what you are actually doing when you are multiplying the *top* by 2 and the *bottom* by two, and don't, don't get into the habit of just multiplying the top or just putting two-thirds times two (writes $2/3 \times 2$) because that's wrong (puts a cross over $2/3 \times 2$). Ok. So we have four over six.

The teacher also used the opportunity to emphasize the fact that when finding equivalent fractions what the students were actually doing was multiplying by 1. The teacher also used the word over when they referred to the fraction 4/6 as "four over six".

To help students remember which part of the fraction is the numerator and which part is the denominator, some teachers pointed to the letter *u* in the word n*u*merator which they said should help students when they think of the word up and to the letter *d* in the word *d*enominator which was also found in the word down. The word *north* was also mentioned as a way of trying to help students remember which part of a fraction was the numerator. These examples illustrated an effort by the teachers to relate what was being learned to the knowledge that students already had even when the knowledge was not necessarily mathematical.

An effort to relate fractions to contexts with which students are familiar with was demonstrated by the use of the words *nanny* and *neighbor* in place of the word numerator. Together with these two words, the word *dog* was used in place of denominator. Later we will see how these words were used in an effort to help students remember an algorithm for converting fractions into decimals.

While the use of natural language such as the words top, bottom, north, nanny, neighbor, and dog might help students relate to contexts students are familiar with, it might create or reinforce the perception that the numerator and denominator of a fraction are two separate numbers. Similar arguments can be made about the use of the word over such as in “2 over 5” for the fraction 2/5 and use of the phrase out of as in ‘6 out of 12” for the fraction 6/12. Use of the word over was observed in teacher T1, T2 and T7’s classes. A student in teacher T1’s class referred to the fraction 9/10 as “nine over ten” and the teacher referred to 8/100 as “eight over hundred”. The use of the phrase “out of” seemed to stem from the use of pattern blocks as seen in teacher T1' class when they were talking about covering the star part of a hexagon with rhombi and triangles, a highly contextualized situation. The whole hexagon could be covered by 12 rhombi or 24 triangles and the star could be covered by 6 rhombi or 12 triangles. The teacher referred to the fractions 6/12 and 12/24 as "6 out of 12" and "12 out of 24" respectively. Teacher T6 also used of the phrase "out of" several times when they were discussing how to convert among fractions, decimals, and percents.

Three teachers (T2, T5, & T7) used the phrase *wonderful one* to refer to fractions that were equivalent to the whole number 1, that is, fractions in which the numerator and the denominator were the same, for example the fractions 1/1, 12/12, and 107/107. The phrase *wonderful one* was used by teachers who said two fractions were equivalent if they share the *wonderful one*, in other words the *wonderful one* can be useful when one wants to check if two fractions were equivalent. The phrase was also used when teachers said you can obtain an equivalent fraction by multiplying by a *wonderful one*. The following transcript shows how the phrase *wonderful one* was used by one teacher.

T: Okay. (*Calls upon student*), what is a *wonderful one*?

S: A fraction equivalent to one.

T: A fraction equivalent to one. Excellent. Right. A fraction that is equal to one is a *wonderful one*. For example, six over six. Is a fraction? Yes? Is equal to one? Yes. For example, what about ten thousand over ten thousand.

S: Yeah.

T: Is it a fraction?

S: Yeah.

T: Yes. Absolutely. Equal to one? Absolutely. Okay. So we've talked about fractions. *Wonderful one*. Now, how do I tell if two fractions are equivalent? Or equal? Now, how would I tell that? Number four? (*Chooses another number*)

S: Ummm. . . I put if they share a factor.

T: Okay, if they share a factor. A little more explanation from that? What do you mean? Let me give you an example and see if that holds true. Let's see I have one-half and two-fourths. You said they share a factor. What do you mean by "they share a factor"?

S: Well . . .

T: Do you mean by going from one to the other?

S: Yeah.

T: Okay. Okay. The *wonderful one*. If you're going from one fraction to another fraction, of course they are going to share the *wonderful one*.

In the discussion about the *wonderful one* the teacher and students talked about the *wonderful one* being a *fraction* equivalent to one. Earlier on I mentioned how some teachers defined a fraction as part of whole. Saying the wonderful one is a fraction does not seem to be in line this definition of a fraction. In fact the following question arises; is a *wonderful one* a fractions or a mixed number?

Half numbers, *parts*, *pieces*, and *fractional parts* were words or phrases which were used by teacher T6 to refer to numbers to the right of a decimal comma. The teacher referred to the decimal part of a number as the *half-side*. Use of the prefix half seems to suggest the numbers to the right of a decimal comma are not actual numbers, when in fact a number such as 0.25 which I guess would be considered a half-number by teacher T6 is not deficient in any way. The use of the phrase half-number also opens up the possibility that a number could be say a quarter-number, a three-quarter number or any other name that students can possibly come-up with. The same teacher talked about "getting rid of the decimal" when moving from a decimal to a percent. While that may be true for some decimals, the same assertion would not be true for example when one writes say the decimal 0.5648 as 56.48%.

Teacher T9 pointed out that saying for example "point six" when referring to the decimal 0.6 was not the correct way of saying it. The teacher went on to tell their students that "if you read out a decimal correctly it tells you how to write it as a fraction". Reading-out the decimal 0.6 as "six-tenths" helped students to write the equation 0.6 = 6/10. Another example that they worked on was 15.64 which the teacher said should be read as "fifteen and sixty-four hundredths" and is therefore 15 64/100 when written as a mixed number.

Is 2/4 Really Bigger Than ½?

The words *bigger*, *larger,* and *smaller* were used by five teachers in different contexts. Three teachers used at least one of the words to characterize equivalent fractions obtained by either multiplying or dividing the numerator and denominator of a fraction by the same whole number. In one class the teacher talked about using the *wonderful one* "to multiply and make fractions bigger." Two of these three teachers also used the words to define an improper fraction while one of these two teachers used the word larger in the context of comparing fractions, in particular when they posed the question; which is larger 13/50 or ¼? The fourth teacher only used the word larger when they defined an improper fraction as a number in which the numerator is larger than the denominator. The fifth teacher only used the words smaller and bigger when they talked about getting smaller or bigger numbers and not smaller or bigger equivalent fractions. Use of the words bigger, larger, and smaller in the context of equivalent fractions can be confusing to students given the fact that the words were used by some teachers in the contexts of comparing fractions and defining improper fractions. Using such words may create a perception that one of the fractions is less than or greater than the other fraction when in fact the two fractions are equal.

Other Characterizations

The word *quotient* was used in place of improper fraction when one teacher asked students to write a mixed number as a quotient. Although the word *mixed fraction* was used in some literature, one teacher (T3) did not allow students to use it in place of the word mixed number.

Mathematical Procedures or Algorithms

The words *doubling* and *halving* were used to describe the operations of multiplying or dividing a fraction by the multiplicative identity 2/2 while the word *tripling* was used by some teachers to refer to multiplying by 3/3. Teacher T5 used the word halving when they were discussing the process of reducing 4/6 to 2/3. In teacher T1's class there was a discussion of getting equivalent fractions by doubling or tripling.

> T: Equivalent fractions, fractions that name the same amount. Now, what if I don't have some pattern blocks or a hexagon or something and I just wana know what would be an equivalent fraction to equal three-fifths (Writes 3/5 on the board). How can I generate an equivalent fraction just from that? Just if I give you some numbers (Calls upon student)
>
> S: double
>
> T: Ok that's one way. I could double. Double means to do, means to what? Ok that's one way (Writes on the board, see Figure 4). Without doubling, what's another way I could generate an equivalent fraction? (Writes 3/5 on the board and calls upon student)
>
> S: Tripling
>
> T: Ok. I sure could. I could say times three. But what am I doing each time?

S: Multiplying

S: You are doing the same thing as, ok you are multiplying the same thing with the top and the bottom

T: Top and bottom means the numerator and the denominator.

In the discussion that followed the above transcript, the teacher went on to say "you have to make sure the numerator and the denominator have the same thing in common. You can't do one without doing the other." In line with this, the teacher asked the class why 2/9 is not equal to 4/27. A student responded and said, "You only doubled the numerator and tripled the denominator." The words doubling and tripling were used in two different contexts and that occurred within a space of less than two minutes. Using the words in the first context (multiplying 3/5 by 2/3 and 3/3 respectively) is not correct because they were in fact multiplying by one in both cases. The use of the words in the second context (doubling the 2 and tripling the 9 in the fraction 2/9 to get 4/27) was correct.

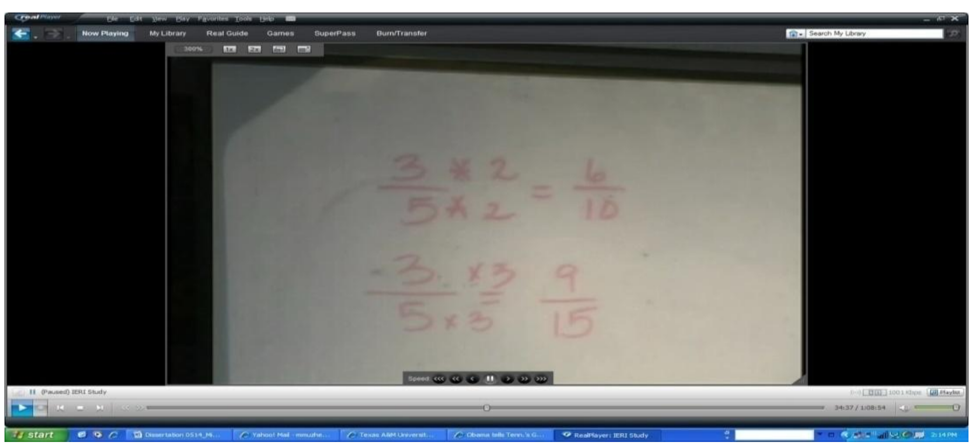


Figure 4: Obtaining an equivalent fraction.

Mathematical Colloquials

The phrase *cowboy rule* was used to describe a procedure of converting a fraction into a decimal. In one of the classes in which this word was used, the teacher created a diagram which they referred to as a *conversion triangle* (see Figure 5). The three corners of the triangle were labeled with the words fraction, decimal, and percent. The procedure of converting decimal to a percent was called *2 to the right*, while the procedure of changing from percent to fraction was called *out of 100*. The procedures for converting a fraction into a percent and converting a decimal into fraction were also vaguely expressed as *common denominator* and *place value* respectively.

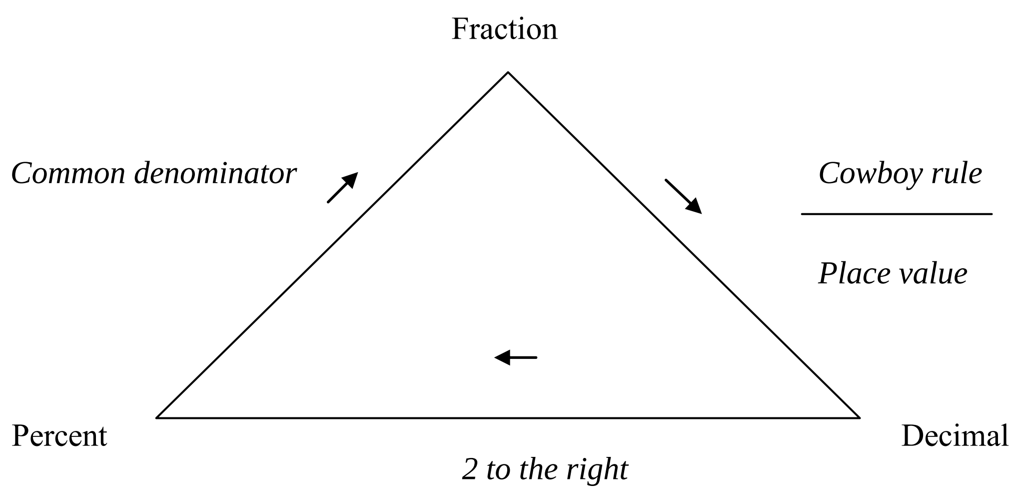


Figure 5: Conversion triangle.

Language and Implied Magnitude

The words *reducing, simplifying,* and the phrases *making smaller* (T5) and *breaking down* (T2 & T7) were used interchangeably to describe the process of obtaining an equivalent fraction by dividing numerator and denominator by the same whole number. The word reduce was used in the two different contexts, the first being they reduce fractions to check equivalence and the second, reducing to find equivalent fractions. Discussions about checking if two fractions were equivalent often brought out cross multiplying as a way of checking equivalence although teachers in the classes where these discussions were held did not allow students to use the method. Their argument was that students should only use techniques that they understood how and why they worked. The use of the phrases *making smaller* and *breaking down* seemed to emanate from the fact that teachers often worked with figures or manipulatives such as fractions strips that they subdivide to get equivalent fractions. It makes sense to use the phrases in that context because the parts in fact get smaller when they are subdivided. This was witnessed when teacher T7 used the phrase *cut-up* when they were subdividing a figure.

Obtaining an equivalent fraction by multiplying numerator and denominator by a whole number (*wonderful one*) was characterized as *building-up* (T2), *expanding* (T2) or *making the fraction bigger* (T2). Use of such words can create the perception that the fractions obtained are not equivalent to the original fraction when in fact what they are doing is multiplying by one. It is important, therefore, to emphasize multiplication by one and simply talk about obtaining an equivalent fraction.

Connections with Student Prior Knowledge

While discussing how to find equivalent fractions by reducing or simplifying, teachers T5, T6, and T9, who by the way were all using different textbooks, made connections to topics they had discussed in earlier lessons, in particular they talked about the greatest common factor and how it can be used to simplify fractions. Teacher T5 used *factor trees* and teacher T9 the method of *cancellation* just as the textbook did. Figure 6 shows examples of how factor trees were used to factorize the numbers 6 and 10. The same figure shows how after writing 0.6 as 6/10 the book used cancellation to obtain the fraction 3/5. Looking closely in the picture you can see that the numbers 6 and 10 were crossed-out in the line where it says “divide the numerator and denominator by the GCF, 2.”

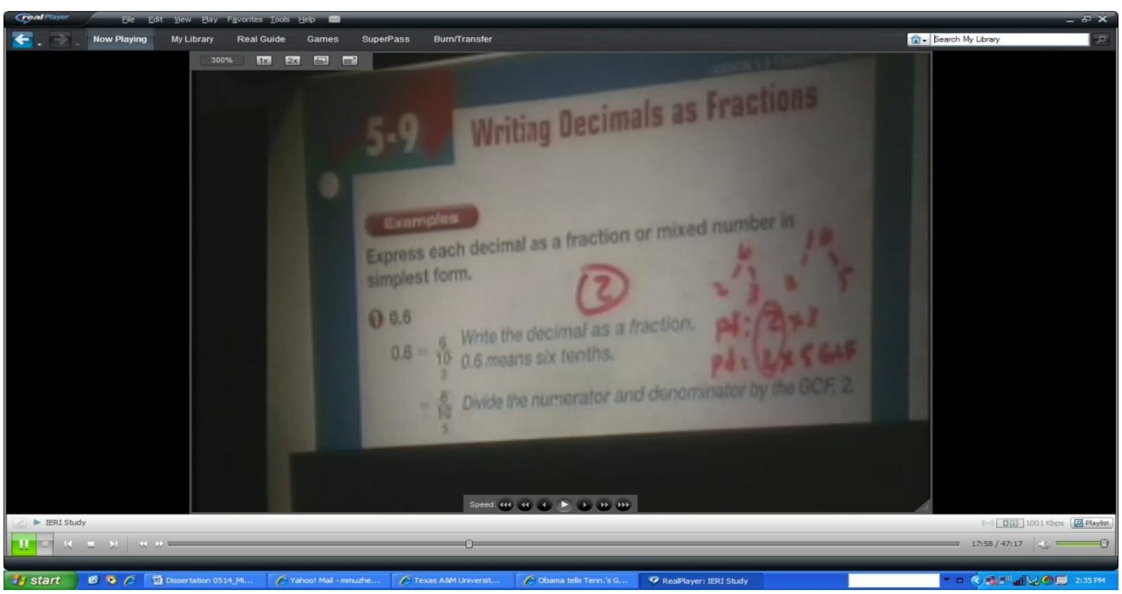


Figure 6: Factor trees and cancellation method.

Natural Language and Algorithms or Procedures

In an effort to help students remember that they should either multiply or divide both numerator and denominator by the same whole number if they want to find an equivalent fraction, one teacher added, “if a boyfriend goes to the movies, then the girlfriend has to go too.” In most instances teachers did not mention the fact that when they multiply or divide the numerator and the denominator by the same number what they in fact are doing is multiplying or dividing by one or the multiplicative identity 1. Most teachers only emphasized *doing the same thing* on the numerator and the denominator.

The phrases *check method* and check mark were used to denote the procedure of converting a mixed number into an improper fraction. Teachers T3 and T12 who used the phrases actually drew the check mark on the transparency every time they showed how to convert a mixed number into an improper fraction. Figure 7 shows two examples, the fractions 2 ¼ and 2 2/7 with check marks written over the numbers. The check mark was used to illustrate how to convert a mixed number to an improper fraction. The numerator of the improper fraction is the product of the whole number part of the mixed number and the denominator of the fractional part plus the numerator of the fractional part. The denominator of the improper fraction is the same as the denominator of the fractional part of the mixed number. So for example, to convert 2 ¼ to an improper fraction, one gets the numerator by multiplying 2 and 4 and then adding 1 to get 9. The denominator of the improper fraction is 4 which is the same as the denominator of 1/4. This process can be expressed as $2\frac{1}{4} = \frac{(2 \times 4) + 1}{4} = \frac{9}{4}$.

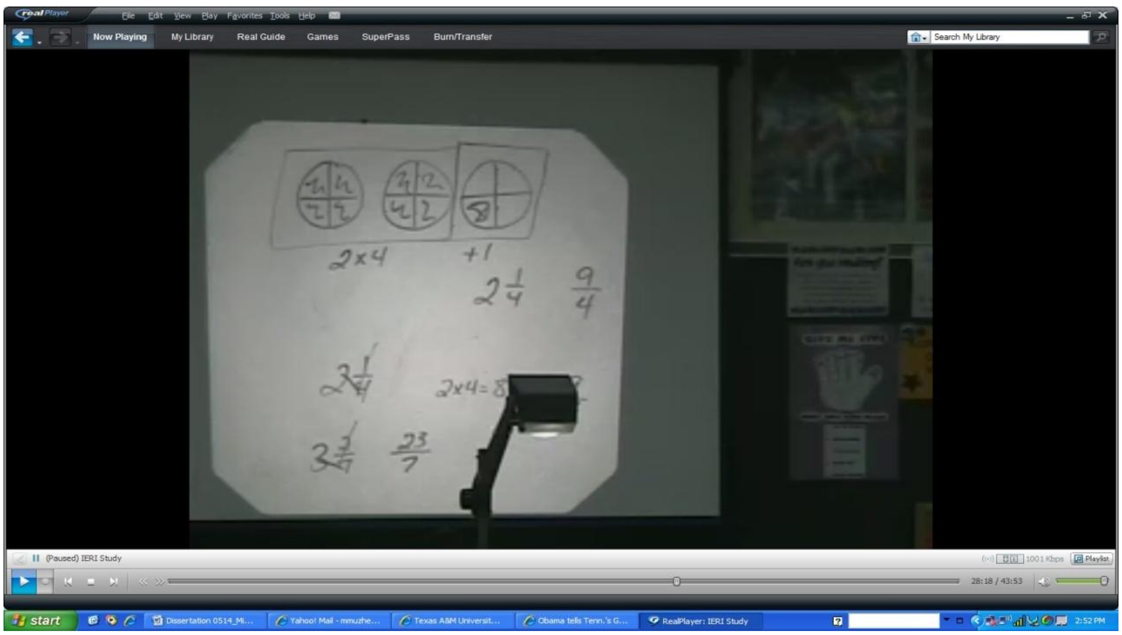


Figure 7: Check mark method.

Some teachers said you convert an improper fraction into a mixed number by *dividing*. As seen earlier the words *nanny* and *neighbor* were used in place of the word numerator and the word dog was used in place of denominator. To help students remember how to convert an improper fraction into a mixed number or a fraction into decimal, teachers T6, T9, and T12 called the square root symbol the *house*. The following is a discussion that went on in teacher T6's class and like teachers T9 and T12, the teacher was relating what was being learned to contexts that students were familiar with.

T: Let's talk about if for a second. They have given us one here, but I am gonna make-up my own. Let's try ummm….say for instance I have … two-fourths, okay. Alright let's talk about it for a second. What do we have? We know that … this is the what? (*Points to number written on transparency*)

S: Numerator

T: Numerator and this is the what? (*Pointing to number written on transparency*)

S: Denominator

T: Denominator. Sometimes we call the numerator the who?

S: The nanny

T: The nanny or the what?

S: Neighbor

T: The neighbor, and we call the denominator the who?

S: The dog

T: The dog. So when we set-up our division problem what does it look like? Who goes out who goes inside?

S: The neighbor goes inside

T: The neighbor lives in the house and the dog lives where?

S: Outdoors.

T: Outdoors. So the dog is who? What number?

S: Four

T: Four and the nanny is who?

S: Two

T: Two. Okay. How many times can four go into two?

Visual Representations

Among the visual representations used by teacher were equations, pictures, diagrams, figures, hundreds grid, tables, charts, and manipulatives such as pattern blocks, fraction strips, and dice.

Missed Opportunities and Idiosyncratic Representations

Although all the teachers explicitly used the equal sign to relate equivalent fractions or fractions, decimals, and/or percents at some stage, there was widespread use of equations which I classified as being idiosyncratic representations. There were numerous occasions where teachers could have used the equal sign symbol, but did not, choosing instead to list or write numbers next to each other without relating them. Thirteen (13) of the sixteen teachers used an equation missing an equal sign at least once. To illustrate how wide spread the issue was, I will point to T5 who did not use the equal sign in at least fifteen (15) instances where they could have done so. Examples included writing say ½ 2/4 3/6 50/100 50%, 5/3 1 2/3, and 78/100 0.78 78% without an equal sign between the numbers. Figure 7 illustrates how one teacher was not relating mixed numbers and improper fractions by leaving out the equal sign. In the figure, the teacher did not relate 2 ¼ and 9/4 and also 3 2/7 and 23/7. There were two teachers (T2 & T4) who at times instead of using the equal sign chose to use a forward arrow.

Although the majority of teachers that were part of this study explicitly used the equal sign to relate equivalent fractions, the equations that some of them wrote were classified as being idiosyncratic due to the fact that some teachers (T1 & T7) wrote numbers with which they were multiplying or dividing with above and below the equal sign. Figure 8 shows screen-captured examples of how teacher T1 was writing numbers below and above the equal sign. This type of representation was seen as significant in that teachers who were using it emphasized doing the same on the numerator and the denominator and instead did not emphasize multiplying or dividing by fractions that are equivalent to 1. The way the operation was represented seems to suggest that you were operating on the numerator and the denominator separately.

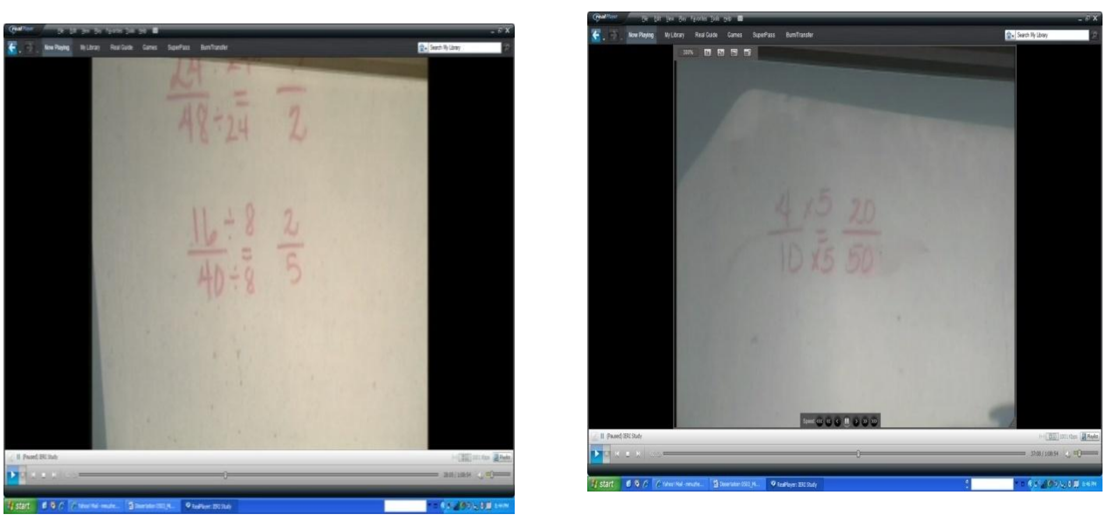


Figure 8: Numbers above and below the equal sign.

One teacher (T11) wrote the numbers with which they were multiplying with or dividing with to get equivalent fractions together with the symbols as subscripts and superscripts. Examples of expressions that the teacher wrote are the following:

$$\frac{1^{\times 20}}{5^{\times 20}} = \frac{20}{100} \qquad \frac{3^{\times 25}}{4^{\times 25}} = \frac{75}{100}$$

Although the teacher got the correct answer in each case, they talked about doing the same thing on the numerator and the denominator and missed the opportunity to emphasize that they were in fact multiplying by 1. Because the 20's and the 25's in the above expressions were written as either superscripts or subscripts it was not clear that they were multiplying by 1 which would had been clearer have the 20's and 25's been written as 20/20 and 25/25, respectively. Using superscripts in the context of finding equivalent fractions can lead to misunderstandings when superscripts are supposed to be interpreted as exponents. Figure 9 shows how teacher (T14) used two equal signs where one equal sign would have been sufficient. The teacher appeared to be treating the numerator and denominator of a fraction as two separate entities. In fact, the teacher omitted the dividing line that would have suggested that they were multiplying by 5/5= 1. A similar representation was seen earlier (Figure 4) when a different teacher did not use the dividing line. Figure 9 also showed the teacher representing the fractions 1/3 and 5/15 with rectangular figures. A close look at the figure drawn to represent 5/15 showed that the rectangular figure was divided into 15 parts with different areas. This highlights the limitations of this type of representation and one hopes the teacher pointed to the limitations of such representations and emphasized the fact that each fractional part needed to cover the same area within the unit region.

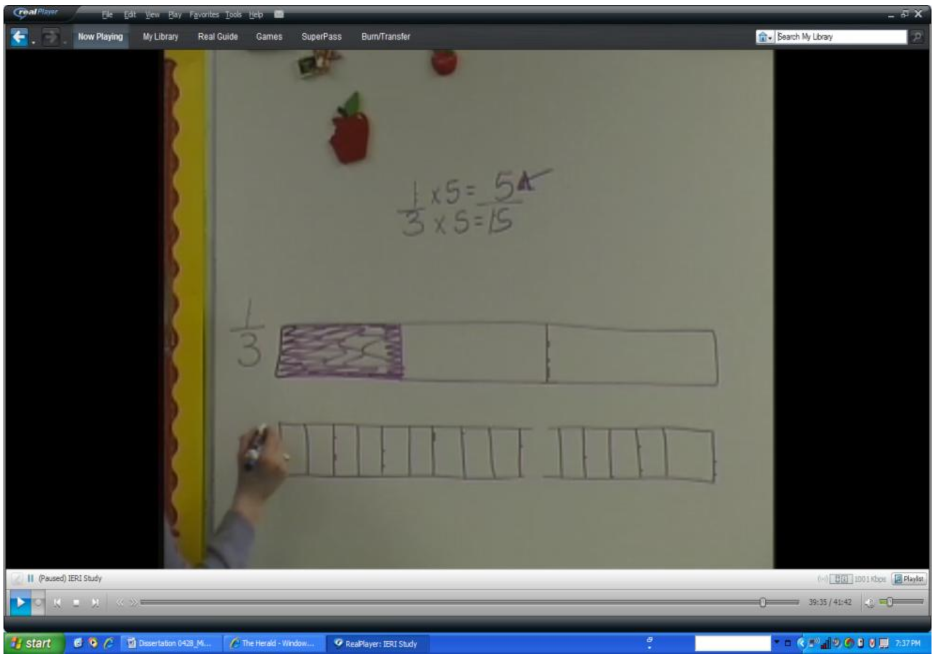


Figure 9: Double equal sign.

Figure 10 shows some of the representations that were used by teacher T9. The first representation that I point to is at the bottom of the picture and showed how the teacher explained how to convert 4 ¾ to an improper fraction. The teacher wrote $4_{(\times)}\,\frac{(+)3}{4}\;\;\frac{19}{4}$ to illustrate they were carrying out the operations (4 × 4) + 3 in the process of obtaining 19/4. Notice that the teacher did not write an equal sign between the numbers. The second thing that the teacher did was represent the mixed number 4 ¾ and the improper fraction with circular figures. The circles used in representations of each of these numbers were clearly not of the same size.

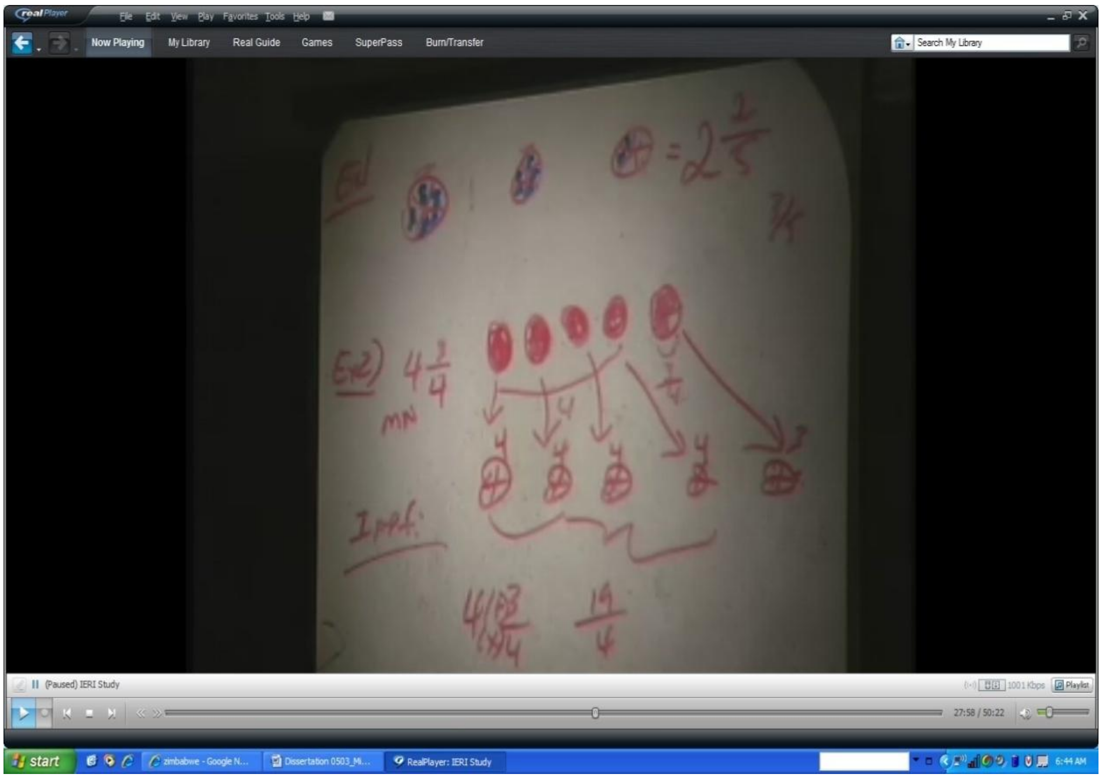


Figure 10: Examples of representations used by T9.

When explaining how to convert a mixed number to an improper fraction, T15 used representations similar to $3_{\times}\,^{12+}1/4 = 13/4$ and $3_{\times}\,^{15+}2/5 = 17/5$. The goal was to show that for example to convert 3 ¼ to an improper fraction you multiply 3 by 4 to get 12 and then add 1 to get the numerator of the improper fraction. In fact the teacher said to convert a mixed number to an improper fraction you *multiply and add* to convert a mixed number to an improper fraction *divide and subtract* since the two operations are inverse operations.

Pattern Blocks and Fraction Strips

Pattern blocks were used by teachers T1, T5, T8, T12, and T13. They were used for finding equivalent fractions by covering part of a hexagon with rhombuses and triangles. The same teachers also used the blocks to represent mixed numbers and to convert mixed numbers to improper fractions or vice-versa. Teacher T14 was the only teacher who used actual candy bars during a lesson. The other types of pattern blocks that T1 used were called units, rods, and quilts. These were used to represent fractions and decimals. The quilt which had 100 squares was taken as the whole which meant that a unit was 1/100 of a quilt and a rod (made- up of ten units) was 1/10 of a quilt. So the number 0.45 would be represented by 4 rods and 5 units.

Only seven of the teachers used fraction strips when they were discussing converting among fractions decimals and percents. Teacher T3 used fraction strips when they were discussing how to convert improper fractions into mixed numbers. In particular they gave each student five fractions strips which they called fake candy bars and asked the students to split them among three people. Teachers T7, T10, T12 and T14 used fraction strips to find equivalent fractions. With multiple fraction strips of the same length, but divided differently (halves, thirds, fourths, etc) printed on one page the class had to label and then find equivalent fractions by reading-out numbers that were vertically aligned. So for example, ½ would be aligned with and therefore equivalent to the numbers 2/4 and 6/12 in the fourths and twelfths strips respectively. On the other hand, teachers T4 and T16 used fraction strips to convert fractions into decimals. They had several fraction strips on one page and at the bottom was the hundredths fraction strip. To convert a fraction into a decimal they would first estimate it with a fraction with a denominator of one hundred by aligning and then once they got the denominator as one hundred it was easy to express the fraction as a decimal.

Geometric Figures

The most commonly used visual representations were geometric figures that each of the sixteen teachers used in at least one context. Circles and rectangles were the most commonly used, although teachers also used hexagons, trapezoids, and triangles. Figures were used to present fractions (part of a whole) by shading part of the figure. Teachers also used figures to demonstrate equivalence of two or more fractions. This was done mostly by either covering part of the shape with pattern blocks of different shapes or by subdividing a figure representing a fraction. None of the teachers addressed the limitations of some of the representations that they were using. An example was when T1 asked a student to demonstrate that the fractions 4/7 and 8/14 were equivalent. Because the teacher used circles in earlier examples, the student spent some time struggling to represent the fractions using a circle after which she asked the teacher if the figure had to be a circle. Instead of seizing the opportunity to point to the fact that it is not easy to represent the fraction 4/7 or any fraction with an odd denominator with a circular figure, the teacher just said “it takes time.” It was much easier for the student to represent the fractions using a rectangle.

Figures were also used to represent mixed numbers and to illustrate how one can convert a mixed number to an improper fraction. Only one teacher (T15) made an effort to emphasize the fact that the figures that one uses to represent a mixed number should be the same size and shape, although they used the word congruent which is not exactly correct because congruent shapes do not necessarily have the same area. While watching several teachers use figures to convert mixed numbers to improper fractions, it became apparent that some teachers had a hard time convincing students why the denominator of a mixed number is the same as the denominator of the fractional part of a mixed number. So for example given the mixed number 2 ¼ how do teachers convince students that the improper fraction is 9/4? One way some teachers achieved this was to divide each of the two shapes (wholes) representing the number 2 in the mixed number 2 ¼ into four equal parts. This meant they could then write each of the whole numbers represented by the two figures as 1 = 4/4. So 2 ¼ = 1 + 1 + ¼ = 4/4 + 4/4 + ¼ = (4 + 4 + 1)/ 4= 9/4. The hundreds grid was used by eight of the sixteen teachers. The grids were mostly used to represent fractions and in discussions about converting fractions into percents. Although converting fractions whose denominators are 100 into percents does not really require using the hundreds grid, the teachers used such examples. They could have illustrated how to use the hundreds grid to convert fractions with denominators other than 100 into percents, a process that requires understanding a fraction as part of a whole or as part of a group.

Teachers T2, T4, T6, T7, T11, and T14 used tables from the textbook that had cat characteristics and data on the amounts of money the cat owners were willing to pay in the event their cats were kidnapped. This real life data together with the hundreds grid was used to teach how to convert among fractions, decimals, and percents. Teachers T1, T2, T5, and T11 used tables which had at least three columns, one for fractions, one for decimal and another one for percents. The same teachers also used place-value charts on which they wrote decimals that they wanted to converted to fractions. While the use of tables and place-value charts can help students easily visualize the numbers, the equal sign symbol could not be used to express the relationships among fractions, decimals, and percents and, therefore, represented a missed opportunity to explicitly use the equal sign.

Other Visual Representations

Teacher T15 was the only teacher who explained how to convert repeating decimals to fractions using the method illustrated in Figure 11. After writing the fraction 77/100, the teacher explained that the denominator does not quite get to 100 because the decimal is repeating and to get as close as possible to 100, they should subtract 1 from 100 to get 99 resulting in the fraction 77/99.

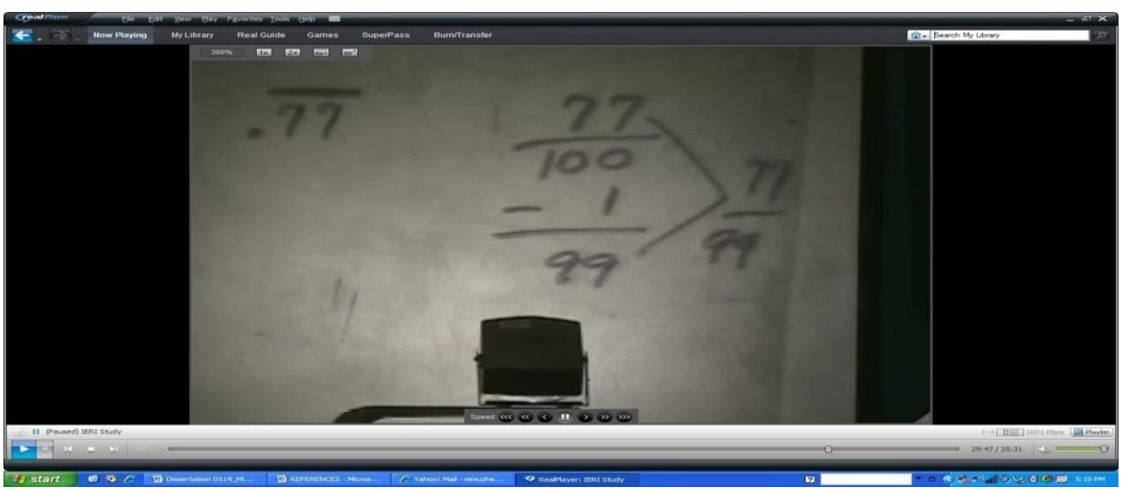


Figure 11: Converting non-terminating decimal to a fraction.

Research Question 2

Having seen in the last section how some teachers were using idiosyncratic representations, this section is concerned with similarities between teacher selected representations and textbook representations. Another goal of this section is to talk about how students' representational choices on the number test are influenced by teacher representations. In particular I address the following question;

What is the role of idiosyncratic representational forms in student solutions as evidenced on their post number tests? In particular,

a) To what extent are the verbal and visual representations of fractions and corresponding mathematical procedures used by teachers and students similar to the representations in the textbook?

b) To what extent are the representations used by teachers in the classroom similar to students' representations on the number test?

c) Do student representational choices reflect teacher idiosyncratic representational forms?

d) Do students' representational forms correspond to various degrees of correct solutions?

Coding Reliability

Reliability of the instrument (Appendix B) was assessed by having a second coder recode about 25% of the representations. There was 100% agreement on coding manipulatives and geometric figures, 92% agreement in coding equations, and 96% agreement in coding verbal representations.

Similarities between Teacher Selected and Textbook Representations

Table 5 summarizes how frequent the numbers 0, 1, and 2 were used to code representations used by teachers on the videos. According to the table, 23% of teachers' representations were coded 0, eighteen percent (18%) were coded 1, and 59% were coded 2. This means about a quarter of all the representations that all the teachers were using were different from textbook representations while more than half of the teachers' representations were adopted from textbooks.

Table 5

Codes

Code	Frequency	Percent	Valid Percent	Cumulative Percent
.00	202	23.0	23.0	23.0
1.00	158	18.0	18.0	41.0
2.00	517	59.0	59.0	100.0
Total	877	100.0	100.0	

Table 6 shows the extent of similarity of teachers' representations and textbook representations. These data were obtained by finding the means of codes. The first column contains data obtained by averaging all codes of representations used by each teacher. The second, third, fourth, and fifth columns contain data obtained by averaging of codes of representations that belong to the same class, namely verbal representations, geometric figures, equations, and manipulatives. The last two columns of the table contain means calculated for the whole sample of teachers and their standard deviations.

Table 6

Extent of Similarity between Teachers' and Textbook Representations

Teacher	Overall mean	Verbal	Figures	Equations	Manipulatives
T1	1.3961	0.5294	2.0000	1.5000	1.8750
T2	0.7157	0.4545	1.4000	1.6667	-
T3	1.3421	1.0000	2.0000	1.3158	2.0000
T4	1.8800	2.0000	2.0000	1.8421	2.0000
T5	1.5391	1.0357	2.0000	1.6102	1.5000
T6	1.0208	0.7692	2.0000	1.0714	-
T7	1.2360	0.5556	2.0000	1.5882	-
T8	1.8214	1.0000	2.0000	2.0000	2.0000
T9	1,3617	1.1154	2.0000	1.5333	-
T10	1.3636	0.5455	-	1.7500	2.0000
T11	1.6250	2.0000	2.0000	1.2500	-
T12	1.3333	0.3636	2.0000	1.5789	2.0000
T13	1.8929	2.0000	2.0000	1.7000	2.0000
T14	1.5789	1.0000	2.0000	1.7222	1.5000
T15	1.5800	0.8000	2.0000	1.6897	2.0000
T16	1.3636	2.0000	-	1.1250	2.0000
Mean over all teachers	1.3603	0.7103	1.9444	1.5685	1.8431
Std. Deviation	0.8318	0.9406	0.3302	0.5551	0.4636

Over all teachers verbal representations deviated from textbook representations the most with a mean of 0.7103 and standard deviation 0.9406. On the other hand, teachers were mostly using geometric figures and manipulatives as suggested in the textbook. The means for these two classes of representations were 1.9444 and 1.8431respectively with standard deviations 0.3302 and 0.4636 respectively. An overall mean of 1.5685 with standard deviation 0.5551 on equations can be explained by the fact that teachers' were generally making small modifications to equations used or suggested in the textbooks that they were using. The overall means show that teachers T4, T8, and T13 mainly used representations that were suggested by the textbooks they were using. Teacher T2 with an overall mean of 0.7157 deviated from textbook representations the most.

To study the effect of each class of representations on the overall mean of each individual teacher, I calculated the means for each class of representations. In columns four and six means mostly equal to 2 or close to 2 suggesting that the majority of teachers were using geometric figures and manipulatives as used or suggested in the textbooks. Means of less than 2 for codes of equations can be explained by the fact that teachers were either omitting the equal sign from equations or they used idiosyncratic forms of equations in which they either wrote the numerators and denominators of the multiplicative identities as superscripts or subscripts or they were writing the numbers above and below the equal sign.

Verbal Representations

Teachers T4, T11, T13, and T16 mostly used verbal representations that were found in their textbooks. Examples of textbook verbal representations would be the words numerator, denominator, reduce, lowest terms, and simplify. The rest of teachers tended to mostly use natural language such as the words top and bottom in place of the words numerator and denominator respectively, in an effort to use contexts that students were familiar with and help them remember concepts under discussion. The means also showed that the verbal class of representations had the largest lowering effect on the overall mean for the majority of teachers. As can be seen from the table, not all teachers used manipulatives in their classrooms. Teachers T2, T6, T7, T9, and T11 did not use any manipulatives. What follows are descriptions of how teacher selected representations that were similar to textbook representations.

Only two teachers who were using the same textbook (Lappan et al., 1998) defined a fraction as part of a whole. As seen earlier this definition made sense in contexts in which one starts with a geometric shape that they divided into equal parts, and then talked about part of the shape. The definition excluded numbers in which the numerator was greater than the denominator, implying that when said fraction, it was understood that they are not talking about improper fractions. A different textbook (Billstein et al., 1999) defined equivalent fractions as fractions that name the same part of a whole, which is in line with the definition of a fraction provided by Lappan et al. (1998). A teacher who used the textbook by Billstein et al. gave a definition of equivalent fractions different from the one in the book when they said, "equivalent means equal or it refers to fractions that name the same number." On the other hand, two teachers who were using the textbook by Lappan et al. defined equivalent fractions as fractions or numbers that name the same quantity. The third textbook (Collins et al., 1998) defines equivalent fractions as fractions that name the same number. None of the textbooks used the following words or phrases; top, bottom, bigger, smaller, *wonderful one*, building-up, expanding, breaking-down, nanny, neighbor, dog, house, check method, half-number, half-side, cowboy rule, doubling, halving, or tripling.

Equations

When finding an equivalent fraction using the multiplicative identity, one textbook (Billstein et al., 1999) used a period instead of the multiplication symbol that was used by all but two of the teachers. So for example the book would have ½ = (1.2)/(2.2) = 2/4 instead of ½ = (1 × 2)/ (2 × 2) = 2/4. Only two teachers (T1 & T5) used the period instead of the multiplication symbol. In one class (T1), after the teacher had finished writing a solution using the period, a student asked the teacher what the "dots" in the equation meant. This example highlighted the limitations of using the period instead of the multiplication symbol as some students can confuse it with the period used in decimals.

It is important to point out that only four (4) of the sixteen teachers emphasized the fact that they were in fact obtaining equivalent fractions by either multiplying or dividing by one. The rest were talking about doing the same thing on the numerator and the denominator. One such example was seen when a teacher said that when the boyfriend goes to the movies, then the girlfriend has to go too. The following example illustrated how one teacher who emphasized multiplying by one explained getting a fraction equivalent to 2/3. The teacher started by writing 2/3 = 2/3 × 1 and then they asked students to "dress the one", after which they re-wrote the 1 as 2/2 with which they multiplied 2/3 by to get 4/6.

Manipulatives and Calculator

The majority of teachers used the visual representations suggested by textbook authors. Where the textbook suggested the use of pattern blocks, most teachers gave their students an opportunity to use the pattern blocks. Teachers also referred to tables, pictures, grids and diagrams in the textbook that they either copied and displayed on an over head projector or asked their students to open to the page of the textbook with the representations. Some teachers also used worksheets with the tables or grids copied from the textbook. Teacher T1 only referred to the pictures on pattern blocks in the book and did not hand out pattern blocks to the students as suggested by the book. Teachers T5, T8, and T13 choose to do one activity by asking students to take turns rolling a dice while they displayed pattern blocks on a projector although the book had suggested that student work in groups. Teacher T1 seemed concerned with maintaining control and moving ahead with the lesson, hence they did not let students use pattern blocks in both lessons that I observed. Instead of allowing students use multiple pattern blocks, teacher T5 only handed out only one piece of each of the pattern blocks suggested by the book and had to resort to tracing the outlines of the shapes. Possibly it was a question of not having enough materials to carry out the activity.

Teacher T15 was the only teacher out of the whole group who used a calculator to convert fractions to decimals. The textbook that the teacher was using (Collins et al., 1998) has a section that discussed using a calculator to convert fractions into decimals.

Representation Choices on the Number Test

Sub-questions (b), (c), and (d) will be answered using information contained in Table 7 and Table 8. In addition to using the representations that are mentioned in the tables, students also used geometric figures to represent fractions. The most widely used geometric figure was the rectangle. Each student used a rectangle at least once on the number test.

Table 7

Student Use of Representations on the Number Test

Teacher	No equal sign				Not multiplying/dividing by 1 to find equivalent fractions			
	Number of students	As a % of all students taught by teacher	Total number of instances	Teacher does same thing (Y/N)	Number of students	As a % of all students taught by teacher	Total number of instances	Teacher does same thing (Y/N
T1	35	48	79	N	9	12	20	Y
T2	28	56	81	Y	2	4	5	N
T3	8	47	25	Y	3	18	5	N
T4	9	39	13	N	3	13	4	N
T5	25	68	87	Y	5	14	15	Y
T6	39	55	156	Y	12	17	33	N
T7	14	33	58	Y	3	7	7	N
T8	14	54	40	Y	6	23	26	N
T9	14	33	26	Y	2	5	2	N
T10	4	40	12	Y	0	0	0	N
T11	23	35	42	Y	10	15	26	N
T12	13	76	54	Y	3	18	10	N
T13	8	36	32	N	2	9	10	Y
T14	33	66	105	Y	10	20	26	N
T15	3	20	3	Y	1	7	2	N
T16	10	50	34	Y	1	5	2	N

Table 8

Student Use of Idiosyncratic Representations on the Number Test

Teacher	Number of student using each representations					
	Double equal sign	Forward arrow	Subscripts Or superscripts	Top and/or bottom	Numbers above and below equal sign	Other verbal representations
T1	2	1	5	4	0	r-e-sign(2)
T2	0	0	0	1	0	c-tri(2) c-rule(5) w-one(2) r-e-sign(1)
T3	2	1	0	1	0	
T4	7	1	0	1	3	
T5	1	2	0	0	0	or(1)
T6	5	3	1	5	3	or(7)
T7	0	3	0	0	6	nanny, dog, house(2) or(2) c-tri(1) c-rule(1)
T8	1	0	0	2	2	
T9	0	0	0	2	0	
T10	0	0	0	0	1	
T11	1	1	0	3	8	or(3)
T12	0	1	1	2	0	
T13	2	1	0	1	1	
T14	11	0	0	1	1	
T15	0	0	0	1	0	
T16	0	2	0	2	0	or(3)

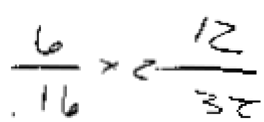
r-e-sign = running equal sign, c-tri = conversion triangle, c-rule = cowboy rule, w-one = *wonderful one*

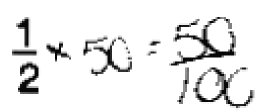
A number of students also explicitly used the equal sign to show relationships among fractions, decimals, and percents. I decided to focus on the representations mentioned in Tables 7 and 8 because almost all of them were considered to be idiosyncratic and, therefore, relevant to the research questions. A number of students were not able to represent 3/5 on a hundreds grid as required for one question on the number test. The wrong responses given by students included shading 15, 20, 25, 30, 35, 40, 50, 53, 70 or 75 squares instead of shading 65 of the 100 squares on the 10 × 10 grid.

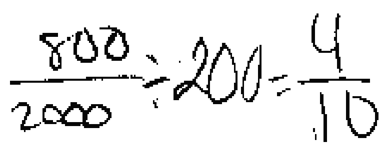
Equations

Table 7 showed there was widespread use of equations in which the equal sign was omitted. In each class at least 20% of the students wrote equations in which the equal sign was missing. The highest number of instances in which students left out the equal sign were observed on the number scripts of teacher T6's students. In all, 55% of the students omitted the equal sign at least once and cumulatively there were 156 instances where the students could have used the equal sign but did not. Table 6 shows that the mean extent to which teacher T6 used equations as in the textbook is 1.0714 which was the lowest mean among all the teachers. Teacher T15's students only exhibited 3 instances in which the equal sign was missing. Although there were at least 3 students from each teacher's group of students who wrote expressions in which they omitted the equal sign, not all teachers wrote such expressions in their classrooms. In particular teachers T1, T4, and T13 used the equal sign consistently to explicitly express relationships among fractions, decimals, and percents. There was no better place to explicitly focus on the equal sign than when discussing relations among fractions, decimals, and percents, in particular when talking about equivalent fractions.

Except for teacher T10's students, at least one student taught by the other teachers multiplied or divided by a number other than one when finding equivalent fractions. Figure 12 showed examples of how some students either multiplied or divided by numbers not equal to 1, but still got fractions equivalent to the one they started with. Although in each case the students managed to get equivalent fractions, the use of the equal sign in the algebraic expressions (b), (c), and (d) is not mathematically correct since ½ × 50 = 25 ≠ 50/100, 800/2000 ÷ 200 = 4/2000 ≠ 4/10, and 52/100 ÷ 2 = 26/100 ≠ 26/50. Although the students did not write equal signs in (a) and in (d) (after they divided by 2 the second time), including the equal sign would have been wrong since 6/16 × 2 = 12/16 ≠ 12/32 and 26/50 ÷ 2 = 13/50 ≠ 13/25.

(a) 

(b) 

(c) 

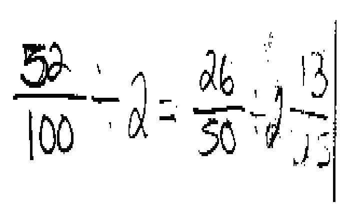
(d) 

Figure 12: Obtaining equivalent while multiplying by whole numbers not equal to 1

While it might seem from the examples in Figure 12 that students can always get away with multiplying or dividing by numbers other than 1 and get equivalent fractions, such a conception can present difficulties when students try to solve a different set of problems. Consider the set of problems represented by the problem $3/8 \times 4 = \square$ or the following item on the number test.

Which of the following is true about 125% of 10?

A. It is greater than 10

B. It is less than 10

C. It is equal to 10

D. Can’t tell

Figure 13 shows a solution that was written by a student trying to solve the above problem. Although the student made the mistake of dividing instead of multiplying 125/100 by 10, it is clear the student divides both numerator and denominator by 10 and ends up with a number less than 12.5. This example illustrated how students who do not multiply or divide by 1 to get equivalent fractions can end-up writing the solution to the above item from the test as $125/100 \times 10 = 1250/1000 = 1.250$ which of course is not correct because $125/100 \times 10 = 12.5$. I point out though that the student was able to select the correct option A, which was amusing because it was not clear how they got the solution from the expression they wrote. It appears the student divided the denominator 100 by 10 to get 10. Upon dividing 125 by 10 the student got 12 and a remainder 25 which they wrote as an exponent.

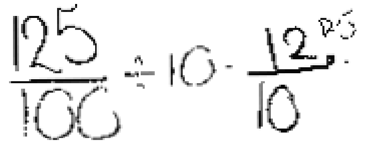
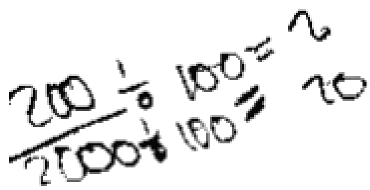
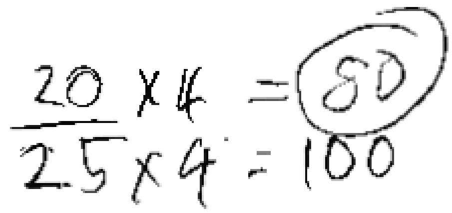


Figure 13: Work shown while trying to find 125% of 10.

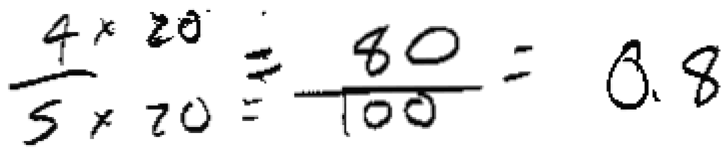
Although only one teacher (T14) was observed using double equal signs (one above the other) in equations where they found equivalent fractions, a total of thirty-two (32) students taught by 9 different teachers used double equal signs on the number test. Figure 14 shows three such examples copied from number test scripts of three different students. Using two equal signs suggested students were treating the numerators and the denominators as separate numbers and not seeing the fractions as single numbers. This is in line with teachers who were emphasizing doing the same thing to the numerator and the denominator instead of emphasizing multiplying or dividing by 1. The largest number of students using double equal signs and taught by the same teacher was 11, and these were teacher T14's students. Teacher T14 never emphasized multiplying or dividing by 1 instead talked about multiplying the numerator and the denominator by the same number and as mentioned above was the only teacher observed using two equal signs where one would have been sufficient. The assertion that students were treating numerators and denominators as separate entities was also supported by the fact that in the examples of Figure 14 there was no dividing line as in the numbers 100/100, 2/20, 4/4, 80/100, and 20/20.



(a)



(b)



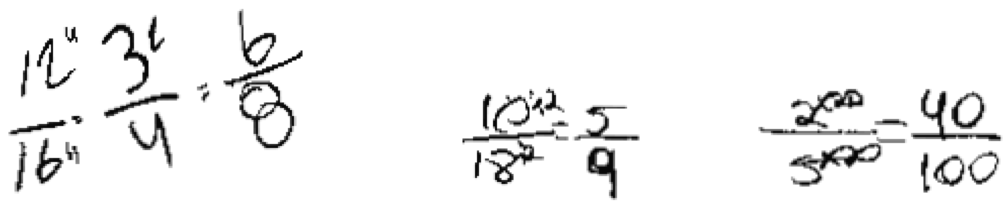
(c)

Figure 14: Use of “double equal signs” on number test.

A total of 16 students used the forward arrow at least once in place of the equal sign. Of the 10 teachers who taught these 16 students, only one teacher (T4) used the forward arrow in place of the equal sign. The only other teacher who also used a forward arrow in place of the equal sign was teacher T2, but none of her students used a forward arrow in place of the equal sign on the number test.

There were 7 students from teachers T1, T6, and T12’s classes who wrote the numbers with which they were dividing or multiplying with to obtain equivalent fractions as superscripts and/or subscripts (see Figure 15). The only teacher who used the superscript and subscript notation in their classroom was teacher T11and none of her students did the same on the number test. Using superscripts and subscripts as in the examples of Figure 15 can cause confusion when students learn about exponents such as

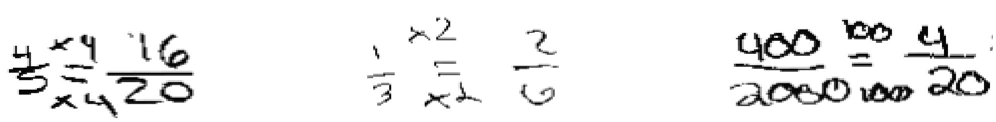
12^4 which should to be interpreted as 12 × 12 × 12 × 12 and not as 12 ÷ 4 as was done by the student in Figure 15 (a).



(a) (b) (c)

Figure 15: Writing numbers and operators as superscripts.

In all, there were 25 students who either wrote numbers or the equal sign and numbers below and above the equal sign as can be seen in Figure 16. Only teachers T1 and T7 were wrote numbers below and above the equal sign when they were teaching. On the number test though students from teachers' T4, T6, T7, T8, T10, T11, T13, and T14 classes were writing equations in which numbers were above and below the equal sign. Hence the only students that could have been influenced by their teacher in using such representations were students from teacher's T7 classes.



(a) (b) (c)

Figure 16: Numbers and operators above and below equal sign.

Verbal representations

The words top and/or bottom showed-up on number tests of 26 students taught by 13 different teachers. The only teachers whose students did not use any of the words on the number test are teachers T5, T7, and T10, although they all used the words top and bottom in their classrooms. The other teachers who used these words top and bottom are teachers T2, T3, T9, T12, T14, and T15. Therefore although teachers T1, T4, T6, T8, T11, T13, and T16 did not use the words top or bottom in place of the words numerator and denominator respectively, at least one student from their classes used the words on the number test. Ironically, teacher T1 who actually took to task students who tried to use the words by asking them to explain what the words meant taught the highest number of students taught by the same teacher who used the words. In addition to using the words conversion triangle, cowboy rule, wonderful-one, nanny, dog, and house on the number test, a total of 16 students used the word *or* in place of the equal sign. There were two students in teacher T16's class who used the word *or* 8 and 9 times each in place of the equal sign. Although there in nothing wrong with using the word *or*, I am of the opinion that using the equal sign especially when discussing the relationships among fraction, decimals, percents demonstrates that one understands that the numbers on either side of the equal sign represent the same quantity although they might look different.

Research Question 3

In this section I report findings on investigations of how representation usage was related to teacher characteristics. Some of the results were obtained from the chi-square test to carry-out hypotheses testing. The question addressed in this section was stated as follows;

Were there differences in the teachers' representational choices (mathematical or idiosyncratic) for fractions and mathematical procedures as a result of teachers' years of experience, level of education, type of certification or other emergent factors based on the quantification of the qualitative data obtained for question 1?

Representation Usage and Lesson Objective

Table 9 shows results of an investigation aimed at determining whether or not representational usage was a function of the lesson objective. The main objective of each lesson was one of the following; (a) finding equivalent fractions (EF); (b) converting fractions into decimals and vice versa (FD); (c) relating fractions, decimals, and percents(FDP); and (d) converting improper fractions to mixed numbers and vice-versa (FMF). The data suggests that in all lessons, verbal usage by all teachers deviated from textbook representations the most, with means of 0.4903, 0.8906, and 0.9750, and 1.0968 for lessons in which the objectives were EF, FDP, FMF, and FD respectively. Use of equations had means of between 1.3000 and 1.6500 across the lessons that can be explained by the modifications that teachers made to textbook equations such as leaving out the equal sign and writing number above and below the equal sign.

Table 9

Mean Representational Usage According to Lesson Objective

Lesson	Representations	Mean	Std. Deviation
EF	Equations	1.6331	.49846
	Figures	1.9412	.34300
	Manipulatives	1.6000	.68056
	Verbal	.4903	.84028
FD	Charts	2.0000	
	Equations	1.6333	.48596
	Figures	2.0000	-
	Manipulatives	2.0000	-
	Verbal	1.0968	1.01176
FDP	Equations	1.3086	.62534
	Figures	1.8667	.50742
	Verbal	.8906	.97780
FMF	Equations	1.6449	.55376
	Figures	2.0000	-
	Manipulatives	2.0000	-
	Verbal	.9750	.99968

Note. EF = finding equivalent fractions, FD = converting fractions into decimals and vice versa, FDP = relating fractions, decimals, and FMF = converting improper fractions to mixed numbers and vice-versa

In line with the discussion on the last paragraph, I investigated whether or not representation usage by the teachers was related to the lesson objective. The following hypotheses were tested:

Ho: Representation usage by teachers is independent of lesson objective.

Ha: Representation usage by teachers is related to lesson objective.

Table 11 shows how the codes 0, 1, and 2 were used to code representations according to lesson objective while Table 10 gives a Pearson chi-square value of 51.134 with corresponding probability $p < 0.001$ which is less than 0.05. Therefore at probability level 0.05, I rejected the null hypothesis and concluded that representation usage and lesson objective were related variables.

Table 10

Chi-Square Tests: Lesson Objectives

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-Square	51.134	6	< .001
Likelihood Ratio	52.690	6	< .001
N of Valid Cases	877		

Table 11

Representation Usage According to Lesson Objective

		Codes			Total
		.00	1.00	2.00	
EF	Count	118	59	178	355
FD	Count	14	22	93	129
	Expected Count	29.7	23.1	76.2	129.0
FDP	Count	46	45	109	200
	Expected Count	46.1	35.8	118.1	200.0
FMF	Count	24	31	138	193
	Expected Count	44.5	34.6	114.0	193.0
Total	Count	202	157	518	877
	Expected Count	202.0	157.0	518.0	877.0

Note. EF = finding equivalent fractions, FD = converting fractions into decimals and vice versa, FDP = relating fractions, decimals, and FMF = converting improper fractions to mixed numbers and vice-versa

This section is concerned with investigating whether or not representation usage and the curriculum, in particular the textbook that the teachers were using were related. To investigate the relationship, the following hypotheses were tested:

Ho: Representation usage by teachers is independent of textbook.

Ha: Representation usage by teachers is related to textbook.

Table 12 shows how the codes 0, 1, and 2 were used to code representations according to lesson objective while Table 13 gives a Pearson chi-square value of 28.530 with corresponding probability $p < 0.001$ which is less than 0.05. Therefore at probability level 0.05, the null hypothesis was rejected. Hence there was a relationship between representation usage and textbooks.

Table 12

Representation Usage and Its Relationship to Textbooks

			Codes			Total
			.00	1.00	2.00	
Curriculum	CM	Count	110	64	171	345
		Expected Count	79.5	61.8	203.8	345.0
	MAC	Count	18	15	64	97
		Expected Count	22.3	17.4	57.3	97.0
	MGMT	Count	74	78	283	435
		Expected Count	100.2	77.9	256.9	435.0
Total		Count	202	157	518	877
		Expected Count	202.0	157.0	518.0	877.0

Note. CM= Connected Mathematics, MAC = *Mathematics Applications and Connections*, and MGMT= *Middle Grades Math Thematics*.

Table 13

Chi-Square Tests: Textbooks

	Value	df	Asymp. Sig. (2-sided)
Pearson Chi-Square	28.530(a)	4	< .001
Likelihood Ratio	28.222	4	< .001
N of Valid Cases	877		

Representation Usage and Teacher Characteristics

Investigations were done to study whether or not representation usage by the teachers was related to (a) highest degree obtained by each teacher, (b) certifications, (c) number of years spent teaching mathematics, (d) number of years teaching mathematics at grade level, (e) number of hours completed on professional development (PD) related to their textbook, and (f) the total number of days spent on the Interagency Education Research Initiative (IERI) professional development.

Fisher's exact test gave the value $p = 0.061$ which is greater than 0.05, so I failed to reject the null hypothesis Ho: Representation usage by teachers is independent of the highest degree obtained by each teacher. On the other hand, tests to investigate whether or not representation usage was related to the teacher's certification gave a chi-square value of 17.498 and corresponding probability $p < 0.01 < 0.05$ while the Fisher exact test calculated probability $p < 0.01 < 0.05$. I, therefore, rejected the null hypothesis; Ho: Representation usage by teachers certified for middle school was the same as representation usage by teachers certified at either elementary school or high school and concluded that representation usage depended on whether a teacher was certified at middle school or not. The teachers in this study were certified at one of the following levels: (a) elementary, (b) middle school, and high school. Table 14 shows that teachers with middle school certification had the highest mean extent of similarity between their representations and textbook representations while teachers with elementary school certification had the lowest mean. The means were 1.86 and 1.33 respectively, implying teachers with middle school certification were mostly using representations suggested or used in their textbooks. In fact when I used both the Chi-square test and Fisher's exact test to test if there were any significant differences in representation usage between teachers certified to teach middle school and the rest of the teachers, I got $p < 0.001 < 0.005$ in both tests, leading to the conclusion that there were significant differences in representation usage between teachers certified to teach middle grades and teachers who are not certified to teach middle grades.

Table 14

Mean Extent of Similarity According to Certification

Certification	Mean Extent	Std. Deviation
Elem	1.3333	.86855
Math	1.3521	.81238
MS	1.8571	.44430
Total	1.5181	.76057

Table 15 gives the mean extents of similarities between teacher selected representations and textbook representations according to the number of years a teacher has been teaching mathematics. Investigations of whether or not representation usage was related to the number of years a teacher had been teaching mathematics or to the number of years a teacher has been teaching mathematics at the 6th grade level gave test statistics of 13.088 and 13.060 respectively. The corresponding probabilities of p = 0.004 and p = 0. 001 respectively, were both less than 0.05. At probability level 0.05, I rejected the null hypotheses that claimed representation usage was not related to either the number of years a teacher had been teaching mathematics or to the number of years a teacher had been teaching mathematics at the 6th grade level.

Table 15

Representation Usage and Number of Years Teaching Mathematics

Number of years teaching math	Mean	Std. Deviation
0-5	1.4019	.80070
6-10	1.5944	.68421
11-15	1.4228	.78936
21-25	1.8929	.31497
Total	1.4732	.76498

Hypothesis testing was also used to investigate whether or not representation usage was related to the number of hours of professional development a teacher had on the textbook they were using. Chi-square test gave the test statistic 17.927 and corresponding probability $p < 0.01$ and Fisher's exact test gave probability $p < 0.01$. After rejecting the null hypothesis at probability level 0.05, I concluded that representation usage was somewhat related to the number of hours of professional development a teacher had on the textbook.

Calculations of means of the extent of similarities between teacher selected representations and textbook representations revealed that teacher(s) who attended a total of 15 days of the IERI professional development had the lowest mean of 1.0208 while the highest mean was attained by the teacher(s) who had a total of 24 days (the highest) on the IERI professional development. Teacher(s) who only attended 2.5 days of the IERI professional development had a mean of 1.8800. Hypothesis testing at probability level 0.05 using the chi square statistic led to the conclusion that the total number of days that a teacher had on IERI professional development was not related to how the teachers used representations with a statistic 2.076 and probability $p = 0.345$.

Research Question 4

This section is concerned with how teacher selected representations can lead to student misconceptions. Given the widespread use of idiosyncratic representations by teachers as seen in the previous sections, I sought to address the following question;

> Can enacted student misconceptions on the number test be linked to idiosyncratic representations of fractions and mathematical procedures used by teachers?

Use of two equal signs in one equation by teacher T14 led to 11 students taught by the teacher using the same type of representation on the number test. One could argue that the use of idiosyncratic representations by the teacher in the classroom led to the creation or reinforcement of the misconception that the numerator and denominator of a fraction were two separate entities operated on separately, hence the need to have two equal sign symbols. On the other hand, some students whose teachers were not observed using double equal sign in equations exhibited the same misconception raising the possibility that no teacher had any role in the creation of the misconception.

There was one incident in which a teacher wrote ½ = 2/4 and ½ = 3/6 and a student asked the teacher if 2/4 was equal to 3/6. As suggested earlier combining and writing the two equations as ½ = 2/4 = 3/6 would probably have helped the student to appreciate the fact that 2/4 = 3/6.

DISCUSSION AND CONCLUSIONS

One of the goals of this study was to explore the different types of representations used by middle school mathematics teachers when they were teaching the concepts of converting among fractions, decimals, and percents. The study reveals how natural language plays a role in student learning. In line with Ashlock's (2006) argument that teachers should not be in a hurry to have students use precise mathematical language, but instead help them make connections with terms and concepts they already know, some teachers in this study use natural language in an effort to relate what is being learned to contexts which students should be familiar with. Examples are when teachers use the words nanny and neighbor in referring to the numerator of a fraction and the words down and dog in referring to the denominator. To help students remember which part of a fraction is the numerator, some teachers use the words up and north while the word down is used as a cue for remembering which part of a fraction is the denominator. These are contexts that teachers feel students are familiar with and would help students remember for example, the algorithm for converting a fraction into a decimal.

Some teachers use the word over and the phrase "out of" as in "two over 4" and "2 out of 4" for the fraction 2/4 which according to research (Siebert & Gaskin, 2006) can result in students treating the numerator and denominator of a fraction as separate entities or mere whole numbers. Just like in using the word over and the phrase 'out of", using the words nanny and neighbor in referring to the numerator of a fraction, and the word dog in referring to denominator of a fraction has the potential to lead to a perception that the numerator and denominator of a fraction are separate entities.

This study also reveals that when finding equivalent fractions, some teachers are operating on the numerators and denominators separately as evidenced by the use of two equal signs where one equal sign is sufficient. Related to this is that fact that some teachers are emphasizing “doing the same thing” on the numerator and the denominator instead of seeing the whole process as multiplying or dividing by one. Using two equal signs, can lead to or reinforce the perception that the numerator and denominator of a fraction are two separate entities on which one needs to operate separately.

Natural language combined with manipulatives and/or geometric figures often lead some teachers and students to use phrases such as building-up and breaking-down when they are referring to the operations of obtaining equivalent fractions by either multiplying or reducing. In line with the use of these two phrases is the use of the phrases “bigger equivalent fraction” and “smaller equivalent fraction”. According to Naiser, Wright, and Capraro (2004), some students think that multiplying a fraction by a multiplicative identity 1 other than 1/1 yields a larger fraction. Using the word bigger in the context of equivalent fractions is, therefore, not only confusing and wrong, but can lead to the creation or reinforcement of the misconception that multiplying a fraction by a multiplicative identity 1 other than 1/1 yields a larger or bigger fraction. A similar argument can be made that using the word smaller can lead to a misconception that dividing by a multiplicative identity 1 other than 1/1 yields a smaller fraction.

Research (Hackett, 2002; Pagni, 2004; Sweeney & Quinn, 2000) has shown that a common misconception among students is the thinking that there are no relationships among fractions, decimals, and percents. By not explicitly expressing relationships among fractions, decimals, and percents using the equal sign, teachers can facilitate the creation of or reinforce a misconception by some students that there are no relationships among fractions, decimals, and percents. Other researchers (Knuth, Stephens, McNeil, & Alibali, 2006; McNeil & Alibali, 2005) have argued that American mathematics lessons (K-12) rarely focus explicitly on the equal sign and its meaning and that limited experiences may explain in part why students in middle grades tend to interpret the equal sign as an operational and not a relational symbol. The current study supports the assertion made by these researchers in that some teachers do not use lessons on equivalent fractions, decimals, and percents to reinforce the meaning of the equal sign as a relational symbol.

Researchers (Siebert & Gaskin, 2006) have argued that teachers need to pay attention to the language that students use when they express their mathematical thinking just because expressions students use have underlying images. Some teachers in this study use and permit their students to use the words doubling and tripling when in fact they mean multiplying by 2/2 and 3/3, respectively. Using the words doubling and tripling has the potential of leading students to the thinking that for example 4/6 is double 2/3, as students in the study by Jigyel and Afamasaga-Fuata'i (2007) thought. Teachers should pay attention to the language they use in the classroom given that research (Clement, 2004) has shown that connections that children make between language and written symbols may differ from connections made by adults.

Although no connection was found between using the words doubling and tripling and writing expressions such as $40/60 \div 20 = 2/3$ and $2/5 \times 20 = 40/100$ by some teachers and students, this would appear to be an extension of doubling and tripling to multiplying or dividing by a whole number to obtain equivalent fractions. Some teachers paid attention to language use by students as was seen when one teacher emphasized that students should not read out decimals using the word point, like reading 0.6 as point-six, but instead say six-tenths, language that made it easier for students to convert a decimal into a fraction. Based on the results in this paragraph, I can make conclusions similar to ones that some researchers (Lannin, Townsend, Armer, Green, & Schneider, 2008) have made that students can lack a deep understanding of algebraic symbols that they write. Teachers must, therefore, move beyond the focus on manipulating symbols to include a focus on the internal meaning ascribed by students to written symbols.

Another goal in this study was to examine the role played by teachers' idiosyncratic representations in influencing students' representational choices. Although students are not required to show all work on the number test, there is evidence that students are adopting some of the representations they saw in class. Examples of such representations include equations with missing equal signs, equations with double equal signs, equations in which numbers are written above and below the equal sign, and equations in which numbers are written as superscripts or subscripts. While the use of such idiosyncratic representations has no bearing on the correctness of solutions on the number test, the concern then becomes, how using such representations can affect related or future learning given the fact that rational number concepts are one of the most important mathematical ideas that students encounter before secondary school (Behr, Wachsmuth, Post, & Lesh, 1984). Writing 4^2 when one really means 4×2 can lead to confusion when students are learning about exponents where 4^2 is supposed to be interpreted as 4×4. While some people can argue that the correct use of representations does not always lead to positive learning outcomes, it is better to use representations correctly than having to “un-teach” or “un-learn” the inaccurate representations that students master. The NCTM (2000, p. 15) states “big ideas encountered in a variety of contexts should be established carefully, with important elements such as terminology, definitions, notation, concepts, and skills emerging in the process.”

In addition to using manipulatives such as pattern blocks and fraction strips, the majority of teachers were using geometric figures to represent fraction and mixed fractions. The hundreds grid was used by half of the teachers to illustrate how to convert a fraction into a percent. Use of the hundreds grid to facilitate converting fractions to percents by teaches in this study relies on the understanding of a fraction as part of a whole, and not as part of a group as suggested to be more beneficial by Zambo (2008). The assertion by Zambo that when using the hundreds grid it might be beneficial to understand a fraction as part of group appears to be supported by the fact that a number of students were not able to answer a question on the number test asking them to color in the fraction 3/5 in a hundreds grid and state what percent they had colored. If students have an understanding of a fraction as part of a group, most likely they would be able to color in 3 out of every 5 squares on the grid and then count 60 colored squares out of 100.

Although the NCTM (2000) says students should consider limitations of the various representations that they use, teachers in this study rarely highlight limitations of the representations that they are using and rarely engage students in discussions about limitations of some of the various representations they are using. Having open discussions can help teachers identify any misconceptions that students might have (Naiser, Wright, & Capraro, 2004). Some teachers are representing fractions with geometric figures that are not divided into parts of equal area and some are representing mixed numbers with geometric figures that are not the same size. According to Ashlock (2006), in representing fractions with geometric figures, the parts do not have to be the same shape, but should have the same area. This is important in that some students tend to associate the denominator of a fraction with the total number of parts, and the numerator with the number of shaded parts even when the geometric figure is not divided into parts of equal area (Ashlock).

Another goal in this study was to study how teacher selected representations were similar to textbook representations. Teachers in this study mostly use manipulatives and geometric figures from their adopted textbooks. The majority of teachers make slight modifications to the equations that are used in the textbooks resulting in idiosyncratic representations. One observation is that when requiring the reader to find equivalent fractions, one textbook asks the reader to list fractions that are equivalent to a given fraction. As a result of the word list, some teachers end-up not using the equal sign and instead use commas to separate numbers that are in fact equal. This is a missed opportunity to reinforce the meaning of the equal sign by explicitly expressing relationships among fractions. Although not wide spread, there is evidence that simply listing numbers without relating them or written expressions such as ½ = 2/4 and 2/4 = 3/6 does not automatically lead to the understanding by students that 2/4 = 3/6.

Teacher chosen verbal representations deviate the most from textbook representations. This is mainly due to the fact that teachers try to relate to contexts that students are familiar with by using natural language. Examples of such verbal representations are the words and phrases nanny, neighbor, dog, north, up, house, wonderful one, cowboy rule, breaking-down, and building-up which are not found in textbooks that the teachers are using, but are used to characterize fractions and procedures or algorithms of converting among fractions, decimals, and percents. As it turned out, some of the words or phrases that teachers use are either imprecise or are not mathematically correct. While some researchers (Sun, 2005) concluded that teachers' representation choices are greatly influenced by textbooks, the same conclusion could not be reached on all classes of representations used by teachers in this study. A conclusion similar to Sun's could only be reached with teachers' use of manipulatives and geometric figures. On the other hand, use of verbal representations by teachers in this study is mostly influenced by the need to relate what was being learned to contexts which students are familiar with and not the textbooks.

Results in this study did not reveal many variations in representation usage, in particular idiosyncratic representations based on the number of years a teacher had been teaching mathematics or teaching mathematics at grade level. This finding is not surprising if one considers a finding by Ball (1988) that teaching itself does not produce the kind of mathematical understanding teachers need to teach mathematics effectively. Although subject matter preparation of teachers is rarely a focus of teacher education, learning about the understandings of mathematics that prospective teachers bring with them to teacher education can help colleges to work with prospective teachers so that they move toward the kinds of mathematical understanding needed to teach mathematics effectively (Ball). The apparent understandings of mathematics that teachers in this study demonstrate can help researchers and universities to work with both in-service and pre-service teachers to equip them the with mathematical understanding needed to teach mathematics.

The fact that representations can be considered as the language of mathematics (Coulombe & Berenson, 2001) and that representations are vehicles for learning and communicating (Friedlander & Tabach, 2001) calls for teachers to use representations that will convey the intended information. Teachers should also teach students using representations in ways that will enable them to communicate mathematically. Research (Cai & Lester, 2005) has shown that representations used by teachers influence the representations their students use, which in turn impacts problem solving. While representation choices of students in this study are influenced by teacher chosen representations, no evidence was found to show that problem solving on the number test was impacted by the representations that students chose.

This study has demonstrates that some teachers use verbal and visual representations that are shown to lead to student misconceptions or representations that are already riddled with errors. In addition to focusing on content and teaching methods, teacher education programs should also focus on student thinking and how teachers can avoid using representations that can negatively impact learning in future.

REFERENCES

American Association for the Advancement of Science (AAAS). (1993). *Benchmarks for science literacy*. New York: Oxford University Press.

Ashlock, R. B. (2006). *Error patterns in computation: Using error patterns to improve instruction* (9th ed.). Columbus, OH: Merrill.

Balfanz, R., McPartland, J., & Shaw, A. (2002). Re-conceptualizing extra help for high school students in a high standards era. *Journal for Vocational Special Need Education, 25*, 24-41.

Ball, D. (1988). *The subject matter preparation of prospective mathematics teachers: Challenging the myths*. East Lansing: Michigan State University, National Center for Research on Teaching.

Ball, D. L. (2003, February). *What mathematical knowledge is needed for teaching mathematics?* Paper presented at the U.S. Department of Education, Secretary's Mathematics Summit, Washington, DC.

Ball, D. L., Hill, H.C, & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, *29*(3), 14, 16-17, 20-22 & 43-46.

Ball, D. L., & Sleep, L. (2007, January). *What is mathematical knowledge for teaching, and what are features of tasks that can be used to develop MKT?* Presentation made at the Center for Proficiency in Teaching Mathematics (CPTM) pre-session of the annual meeting of the Association of Mathematics Teacher Educators (AMTE), Irvine, CA.

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the equals sign. *The Elementary School Journal, 84*, 199-212.

Behr, M., Wachsmuth, I., Post T., & Lesh, R. (1984). Order and equivalence of rational numbers: A clinical teaching experiment. *Journal for Research in Mathematics Education, 15*, 323-341.

Ben-Zeev, T. (1995). The nature and origin of rational errors in arithmetic thinking: Induction from examples and prior knowledge. *Cognitive Science, 19*, 341-376.

Ben-Zeev, T., & Star, J.R. (2001). Intuitive mathematics: Theoretical and educational implications. In B. Torff & R. J. Sternberg (Eds.), *Understanding and teaching the intuitive mind* (pp. 29-56). Mahwah, NJ: Erlbaum.

Billstein, R., Williamson, J., Montoya, P., Lowery, J., Williams, D., Buck, M., et al. (1999). *Middle grades math thematics*. Evanston, IL: McDougal Littell.

Blosser, P. E. (1987). *Science misconceptions research and some implications for the teaching of science to elementary school students*. Columbus, OH: Clearinghouse for Science Mathematics and Environmental Education. (ERIC Document Reproduction Service No. ED282776)

Brown, J., Collins, A., & Duguid, P. (1989). Situated cognition and the culture of learning. *Educational Researcher, 18*(1), 32-42.

Cai, J., & Lester Jr., F. K. (2005). Solution representations and pedagogical representations in Chinese and U.S. classrooms. *Journal of Mathematical Behavior, 24*, 221–237.

Capraro, M. M., Kulm, G., & Capraro, R. M. (2005). Middle grades: Misconceptions in statistical thinking. *School Science and Mathematics Journal, 105*, 165-174.

Capraro, R. M., & Capraro, M. M. (2006, April). *Underlying structures of mathematical representation: A theoretical perspective.* Paper presented at the annual meeting of the American Educational Research Association. San Francisco, CA.

Carpenter, T. P., & Moser, J. M. (1984). The acquisition of addition and subtraction concepts in grades one through three. *Journal for Research in Mathematics Education, 15,* 179–202.

Clement, L. L. (2004). A model for understanding, using, and connecting representations. *Teaching Children Mathematics, 11*, 97-102.

Clements, M. A. (1982). Careless errors made by sixth-grade children on written mathematical tasks. *Journal for Research in Mathematics Education*, *13*, 136-144.

Clotfelter, C.T., Ladd, H. F., & Vigdor, J. L. (2006). Teacher-student matching and the assessment of teacher effectiveness. *Journal of Human Resources, 61*, 778-820.

Clotfelter, C. T., Ladd, H. F., & Vigdor, J. L. (2007). *How and why do teacher credentials matter for student achievement?* Retrieved April 28, 2008, from http://www.nber.org/papers/w12828.pdf?new_window=1

Collins, W., Dritsas, L., Frey-Mason, P., Howard, A. C., McClain, K., Molina, D. D., et al. (1998). *Mathematics applications and connections*. Columbus, OH: Glencoe/McGraw-Hill.

Confrey, J. (1990). What constructivism implies for the teaching of mathematics. In R. B. Davis, C. A. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching of mathematics* (Monograph No. 4; pp. 107-124). Reston, VA: National Council of Teachers of Mathematics.

Cooney, T. J., & Wilson, M. R. (1993). Teachers' thinking about functions: Historical and research perspectives. In T. Romberg, E. Fennema, & T. Carpenter (Eds.), *Integrating research on the graphical representation of function,* (pp. 131-158). Hillsdale, NJ: Lawrence Erlbaum.

Coulombe, W. G., & Berenson, S. B. (2001). Representations of patterns and functions: Tools for learning. In A. Cuoco & F. Curcio (Eds.), *The roles of representation in school mathematics: 2001 yearbook* (pp. 166-172). Reston, VA: National Council of Teachers of Mathematics.

DeBoer, G., Morris, K., Roseman, J. E., Wilson, L., Capraro, M. M., Capraro, et al. (2004, April). *Research issues in the improvement of mathematics teaching and learning through professional development*. Paper presented at American Educational Research Association annual meeting, San Diego, CA.

Ding, M., Li, X., Capraro, R., & Capraro, M. (2007, March). *Do elementary children still interpret the "=" sign as an operator?* Paper presented at the annual meeting of the National Council of Teachers of Mathematics, Atlanta, GA.

Gagné, E. D., Yeckovich, C. W., & Yeckovich, F. R. (1993). *The cognitive psychology of School learning* (2nd ed.). Boston: Allyn and Bacon.

Glaser, R., & Bassok, M. (1989). Learning theory and the study of instruction. *Annual Review of Psychology, 40*, 631-666.

Goldhaber, D. D., & Brewer, D. (2000). Does teacher certification matter? High school teacher certification status and student achievement. *Educational Evaluation and Policy Analysis, 22*, 129-45.

Goldin, G. A. (2002). Representation in mathematical learning and problem solving. In L. D. English (Ed.), *Handbook of international research in mathematics education* (pp. 197-218). Mahwah, NJ: Lawrence Erlbaum Associates.

Goldin, G. A., Kaput, J. J. (1996). A joint perspective on the idea of representation in learning and doing mathematics. In L. P. Steffe, P. Nesher, P. Cobb, G. A. Goldin, & B. Greer (Eds.), *Theories of mathematical learning* (pp. 397-430). Mahwah, NJ: Lawrence Erlbaum Associates.

Goulding, M., Rowland, T., & Barber, P. (2002). Does it matter? Primary teacher trainees' subject knowledge in mathematics. *British Educational Research Journal, 28*, 689–704.

Goulding, M., & Suggate, J. (2001). Opening a can of worms: Investigating primary teacher's subject knowledge in mathematics. *Mathematics Education Review, 13*, 41-54.

Greeno, J. G. & Hall, R. P. (1997). Practicing representation: learning with and about representational forms. *Phi Delta Kappan*, 361-367.

Hatano, G. (2003). A conception of Knowledge acquisition and its implications for mathematics education. In J. Kilpatrick, W. G. Martin, D. Schifter (Eds.). *A research companion to principles and standards for school mathematics* (pp. 197-217). Reston, VA: National Council of Teachers of Mathematics.

Hatano, G., Amaiwa, S., & Inagaki, K. (1996). "Buggy algorithms" as attractive variants. *Journal of Mathematical Behavior, 15*, 285-302.

Heaton, R. M. (1992). Who is minding the mathematics content? A case study of a fifth-grade teacher. *Elementary School Journal, 93*, 153-162.

Hennessy, S. (1993). The stability of children's mathematical behaviour: When is a bug really a bug? *Learning and Instruction, 3*, 315-338.

Jigyel, K., & Afamasaga-Fuata'i, K. (2007). Students' conceptions of models of fractions and equivalence. *Australian Mathematics Teacher*, *63*(4), 17-25.

Johnson, R. B., & Onwuegbuzie, A. J. (2004). Mixed methods research: A research paradigm whose time has come. *Educational Researcher*, *33*(7), 14-26.

Jose, M. (1989). *Hispanic and Anglo students' misconceptions in mathematics*. Charleston, WV: Clearinghouse on Rural Education and Small Schools. (ERIC Document Reproduction Service No. ED313192)

Kamii, C., & Clark, F. B. (1995). Equivalent fractions: Their difficulty and educational implications. *Journal of Mathematical Behavior, 14*, 365-378.

Khoury, H.A., & Zazkis, R. (1994). On fractions and non-standard representations: Pre-service teachers' concepts. *Educational Studies in Mathematics, 27*, 191–204.

Kieran, C. (1981). Concepts associated with the equality symbol. *Education Studies in Mathematics, 12*, 317–326.

Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up: Helping children learn mathematics*. Washington, DC: National Academy Press.

Knuth, E. J., Alibali, M. W., McNeil, N. M., Weinberg, A., & Stephens, A. C. (2004). Middle-school students' understanding of core algebraic concepts: Equality and variable. *International Reviews on Mathematical Education, 37*, 1–9.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education, 37*, 297-312.

Korner, C. (2005). Concepts and misconceptions in comprehension of hierarchical graphs. *Learning and Instruction, 15*, 281-296.

Kulm, G., Roseman, J. E., & Treistman, M. (1999). A benchmarks-based approach to textbook evaluation. *Science Books & Films, 35*, 147–153.

Lampert, M., & Cobb, P. (2003). Communication and learning in the mathematics classroom. In J. Kilpatrick & D. Shifter, Eds., *A research companion to principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics, pp. 237-249.

Lannin, J. K., Townsend, B. E., Armer, N., Green, S., & Schneider, J. (2008). Developing meaning for algebraic symbols: Possibilities and pitfall. *Mathematics Teaching in the Middle School, 13*, 478-483.

Lappan, G., Fey, J. T., Fitzgerald, W. M., Friel, S. N., & Phillips, E. P. (1998). C*onnected mathematics*. Menlo Park, CA: Dale Seymour.

Leinhardt, G. (1988). Getting to know: Tracing students' mathematical knowledge from intuition to competence. *Educational Psychologist, 23(2)*, 119-144.

Lincoln, Y. S., & Guba, E. G. (1985). *Naturalistic inquiry*. Newbury Park; CA: Sage.

Mack, N. K. (1990). Learning fractions with understanding: Building on informal knowledge. *Journal for Research in Mathematics Education*, *21*, 16-32.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education, 26*, 422-441.

McCloskey, M. (1983). *Naive theories of motion*. In D. Gentner & A. Stevens (Eds.), *Mental models* (pp. 229-324). Hillsdale, NJ: Erlbaum.

McNeil, N. M., & Alibali, M. W. (2005). Knowledge change as a function of mathematics experience: All contexts are not created equal. *Journal of Cognition and Development, 6*, 285-306.

McNeil, N. M, Grandau, L., Knuth, E. J., Martha W., Alibali, M. W., Stephens, A. C., et al. (2006). Middle-school students' understanding of the equal sign: The books they read can't help. *Cognition and Instruction 24*, 367–385.

Mertens, D. M. (2005). *Research and evaluation in education and psychology: Integrating diversity with quantitative, qualitative, and mixed methods* (2nd ed.).Thousand Oaks, CA: Sage.

Mewborn, D. (2003). Teaching, teachers' knowledge and their professional development. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics* (pp. 45-52). Reston, VA: National Council of Teachers of Mathematics.

Naiser, E. A., Wright, W. E., & Capraro, R. M. (2004). Teaching fractions: Strategies used for teaching fractions to middle grades students. *Journal of Research in Childhood Education, 18*, 193-198.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

Onwuegbuzie, A. J., & Collins, K. M. T. (2005, February). *A typology of mixed methods sampling designs in social science research*. Paper presented at the annual meeting of the Southwest Educational Research Association, New Orleans, LA.

Onwuegbuzie, A. J., & Leech, N. L. (2005, April). *A typology of mixed methods research designs*. Paper presented at the annual meeting of the American Educational Research Association, Montreal, Canada.

Pagni, D. (2004). Fractions and decimals. *Australian Mathematics Teacher*, *60*(4), 28-30.

Pape, S. J., & Tchoshanov, M. A. (2001). The role of representation(s) in developing mathematical understanding. *Theory into Practice, 40*, 118-125.

Perkins, D. N., & Simmons, R. (1988). Patterns of misunderstanding: An integrative model for science, math, and programming. *Review of Educational Research, 58*, 303–326.

Post, T., Harel, G., Behr, M., & Lesh, R. (1991). Intermediate teachers' knowledge of rational number concepts. In E. Fennema, T. P. Carpenter, & S. J. La (Eds.), Integrating *Research on Teaching and Learning Mathematics* (pp. 177-198). Albany, NY: SUNY Press.

Preston, R. V., & Garner, A. S. (2003). Representation as a vehicle for solving and communication. *Mathematics Teaching in the Middle School, 9*, 38-43.

Putnam, R. T. (1992). Teaching the 'hows' of mathematics for everyday life: A case study of a fifth grade teacher. *Elementary School Journal, 93*, 163-177.

Resnick, L. (1983). Mathematics and science learning: A new conception. *Science, 220*, 477-478.

Rivers, J., & Sanders, W. (2002). Teacher quality and equity in educational opportunity: Findings and policy implications. In L. Izumi & W. Evers (Eds.), *Teacher quality* (pp. 13-23). Palo Alto, CA: Hoover Institution.

Rosengrant, D., Van Heuvelen, A., & Etkina, E. (2005, August). Case study: Students' use of multiple representations in problem solving. In P. Heron, L. McCullough, & J. Marx (Eds.), *Physics education research conference* (pp. 49-52). Paper presented at the American Institute of Physics conference, Salt Lake City, UT.

Sáenz-Ludlow, A., & Walgamuth, C. (1998). Third graders' interpretations of equality and the equal symbol. *Educational Studies in Mathematics, 35*, 153-187.

Sandelowski, M. (2000). Focus on research methods combining qualitative and quantitative sampling, data collection, and analysis techniques in mixed-method studies. *Research in Nursing & Health, 23*, 246–255

Sanders, W. J., & Horn, S. P. (1998). Research findings from the Tennessee Value Added Assessment System (TVAAS) database: Implications for educational evaluation and research. *Journal of Personnel Evaluation in Education*, *12,* 247-256.

Saxe, G. B. (1988). Candy selling and math learning. *Educational Researcher*, *17*(6), 14-21.

Scheuermann, A., & van Garderen, D. (2008). Analyzing students' use of graphic representations: Determining misconceptions and error patterns for instruction. *Mathematics Teaching in the Middle School, 13*, 471- 477.

Siebert, D., & Gaskin, N. (2006). Creating, naming, and justifying fractions. *Teaching Children Mathematics, 12*, 394-400.

Smith, J. P., diSessa, A. A., & Roschelle, J. (1993-1994). Misconceptions reconceived: A constructivist analysis of knowledge in transition. *The Journal of the Learning Sciences*, *3*, 115-163.

Southwell, B., & Penglase, M. (2005). Mathematical knowledge of pre-service primary teachers. In H. L. Chick & J. L. Vincent (Eds.), *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 4, pp. 209-216). Melbourne: Psychology of Mathematics Education.

Stavy, R., & Tirosh, D. (2000). *How students misunderstand science and mathematics: Intuitive rules*. New York: Teachers College.

Stein, M. K., Baxter, J. A., & Leinhardt, G. (1990). Subject-matter knowledge and elementary instruction: A case from functions and graphing. *American Educational Research Journal, 27*, 639-663.

Sweeney, E. S., & Quinn, R. J. (2000). Concentration: Connecting fractions, decimals & percents. *Mathematics Teaching in the Middle School, 5*, 324-328.

Tashakkori, A., & Teddlie, C. (1998). *Mixed methodology: Combining qualitative and quantitative approaches*. Thousand Oaks, CA: Sage.

Thompson, P. W., & Thompson, A. G. (1994). Talking about rates conceptually, part I: A teacher's struggle. *Journal for Research in Mathematics Education*, *25*, 279-303.

Thompson, P. W., & Thompson, A. G. (1996). Talking about rates conceptually, part II: A teacher's struggle. *Journal for Research in Mathematics Education, 27*, 2-24.

Thompson, P. W., Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In J.Kilpatrick, W. G. Martin, D. Schifter (Eds.). *A research companion to principles and standards for school mathematics* (pp. 95-113). Reston, VA: National Council of Teachers of Mathematics.

VanLehn, K. (1990). *Mind bugs: The origins of procedural misconceptions*. Cambridge, MA: MIT Press.

Wayne, A. J., & Youngs, P. (2003). Teacher characteristics and student achievement gains: A review. *Review of Educational Research, 73*, 89-122.

Zambo, R. (2008). Percents can make sense. *Mathematics Teaching in the Middle School, 13*, 418-422.

APPENDIX A

Types of Data and Their Origin

Question type	Research question 1	2	3	4
QUAL	Data s*ource:* videotapes *Data types*: transcripts of selected video clips, images-either recreated or screen captured describing or showing types of verbal or visual representations used by teachers. Appendix B will be used by keep a record of where the data are collected from as well as its description.			*Data sources*: videotapes and student responses on the number test. *Data types*: transcriptions and images demonstrating idiosyncratic representations used by teachers and transcriptions or images describing or showing student representation usage and student misconceptions
QUAN		*Data sources*: videotapes, textbooks and student written answers to the number test. The *numerical data* giving an indication of the extent of differences in representations will be obtained from comparing teachers' representations with textbook representations (see Appendix B) and from counting frequencies in representational usage by students on the number test.		
MIXED			*Data sources:* videotapes, textbooks and MSMP data base in teacher characteristics and student performance. *Data types:* text describing the types of representations (mathematical or idiosyncratic) used by teachers, textual data giving levels of education and type of certification, and numerical data indicating teachers' years of experience.	

APPENDIX B

Locating and Classifying Teachers' Representations of Numbers or Mathematical Procedures and Comparing Teacher's and Students' Representations in the Videos with Textbook Representations

Sighting		Teacher (T) or Student(T)	Type (class) of representation	Idiosyncratic (Y/N)	Extent of similarity:		
Start	End				**0** -teacher's representation totally different from and in a different class from the representation used or suggested in the textbook or a teacher's representation not used in the textbook at all. **1** -teacher's representation and the textbook representation were similar (in the same class), but the teacher had modified their representation, which may or may not have resulted in the two representations conveying different meanings as a result of the teacher's representation being an idiosyncratic representation. **2** -teachers' representation in same class and exactly or almost the same as textbook representations.	Description of 0, 1 or 2	Additional notes

VITA

Michael Tapfuma Muzheve received his Bachelor of Science and Master of Philosophy degrees in mathematics from The University of Zimbabwe in 1996 and 2002 respectively. He also graduated from Texas A&M University in 2005 with a Master of Science degree in mathematics. He entered the Curriculum & Instruction program at Texas A&M University in January 2006 and received his Ph.D. in August 2008. His research interests include mathematics education and graph theory.

Mr. Muzheve may be reached at Department of Teaching, Learning & Culture, Texas A&M University, College Station, Texas, 77840-4232. His e-mail address is mmuzheve@yahoo.com.